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FOR SCHOOLS AND COLLEGES

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PLANE GEOMETRY

PRACTICAL AND THEORETICAL

VOL. I.

PLANE GEOMETRY

PRACTICAL AND THEORETICAL

PARI PASSU

BY

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VOL. I.

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PREFACE

In this Book the aim is to develop practical and theoretical geometry *pari passu*. Appeal is made to the common sense of a beginner, before the *minutiae* are considered, and the opening is with Geometrical Drawing only, so that explanation of the use of instruments is at first given. Throughout the book are found applications to every-day affairs, to games, to Surveying, to Architecture, to Navigation, and to Engineering. Such practical necessities as Mariner's Compass, Latitude and Longitude, Zone Time, Time at Sea are fully gone into. The chapter on small measurements (with special reference to Verniers) is thoroughly practical. Images, and their application to Billiard Table Problems, etc., being among the subjects dealt with, are interesting. A large number of diagrams have been provided; in a book the pictorial argument is the effective one, and memory of a picture is much more real and lasting than a vague recollection of something which was only read.

Reference is made continually to the historical side.

In the Examples the aim has been to give the figures for at least the earliest cases (even the simplest), and often later, too, if clearness was thereby gained. To many of the riders Hints have been added. Geometry is quite a hard enough subject, even if difficulties are reduced to a minimum. The encouragement in success is considerable, and affords the best chance of future development. The last page has been strengthened, points on it can be "pricked off," and it thus affords an unlimited scope for possible questions, allowing the gain of much time in avoiding the tedious preparation of a figure (which too often turns out to be in a bad, and sometimes impossible, position on the page).

Throughout, bookwork and examples are either blank or are preceded by one asterisk (*), or by two asterisks (**), showing what is appropriate to first, second and third readings respectively.

Style is insisted on, and the "hints" (collected at the beginning) are of *constant* application.

In Volume I. the scope is Straight Lines, Angles, Parallels, Triangles, Inequalities, Polygons, Areas (with Pythagoras' Theorem), and Loci. There are chapters too on Symmetry, Converses, Images, Small Measurements and Number of Data. Some of these do not come into the first reading at all.

The examples (over 900) are roughly a third each of Geometrical Drawing, Calculations and Riders. Each class is kept separate though considerable encouragement is made to use one method as a check to another. There are 17 propositions (and three more, only given in outline in the text, are also given fully in the Appendix).

In Volume II. (which covers Circles, Rectangles and Similarity) the same stress is laid on practical applications; but in the 700 odd examples the actual Geometrical Drawing is much less and the theory fully a half. There are 20 propositions (and six more in the Appendix). The same points are available for "pricking off" questions.

A third volume on Solid Geometry will complete the course.

The kind permission of the Controller of His Majesty's Stationery Office to reprint some questions from certain Civil Service Army Examination Papers has been of great value.

My warmest thanks are also due to Mr. C. H. Allcock for placing at my disposal a large collection of riders.

To many others I am grateful for advice, suggestions and help, but most especially to Mr. W. Hope-Jones, whose fertility of imagination and ungrudging assistance in checking the accuracy of every calculation have alone rendered possible the production of this book

V. L. N. F.

June, 1921.

LIST OF CONTENTS

* Should be omitted in a first reading.

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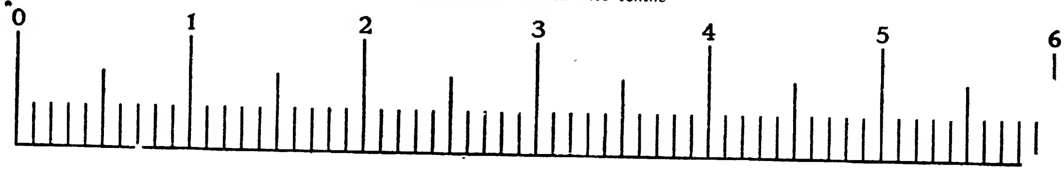
SUMMARY OF PROPOSITIONS

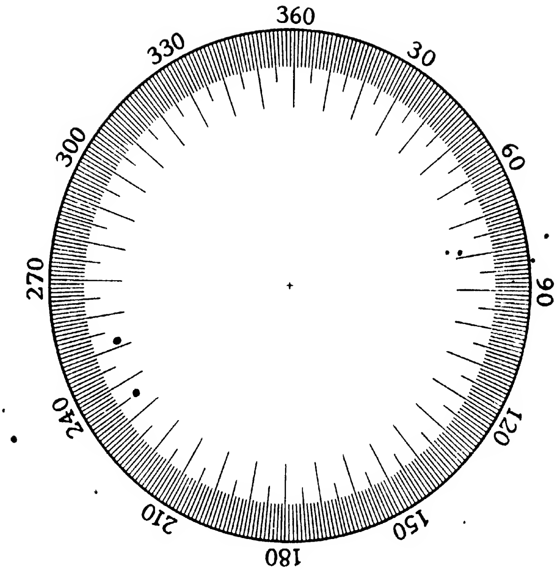
Briefly the Propositions are :

INCHES *divided into tenths*



CENTIMETRES *divided into millimetres*

Paper expands and contracts a little, according to the state of the atmosphere, and it is wiser not to use the above figures for measurement in accurate work.


360
30
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270
300
330

NOTES AND HINTS.

§ 1. **Some reasons for learning Geometry.** Geometry is a science which embodies certain fundamental truths with which an educated person cannot be unfamiliar.

From a utilitarian point of view Geometrical Drawing is essential to the Architect, Surveyor and Engineer.

The facts on which calculations are based are of use to all.

Accurate geometry is the basis of navigation by sea and by air.

Formal geometry gives the type of argument needed in trying to lay out a case soundly, methodically and clearly.

§ 2. **General hints.** The main thing is **clearness,** so that of necessity handwriting should be **easily legible,** and the work **clean.**

§ 3. **Propositions.** It is not a bad plan to write and **underline** the headings *General Enunciation*, *Particular Enunciation*, *Construction*, *Proof*. Separate sentences should begin on separate lines.

§ 4. **Geometrical Drawing.** The main thing is clearness. Distinction between given and final lines and lines merely put in in the construction can be made by full and dotted lines (different colours are useful too). A sharp pencil (H and not HB) is necessary. It can be sharpened with a knife and the lead kept sharp by sandpaper.

§ 5. **Calculations.** Neat freehand sketches are enough, though accurate diagrams afford useful checks.

§ 6. **Riders.** If a proof is not seen soon, assuming what has to be proved and working backwards sometimes affords a clue.

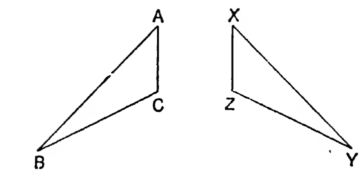
But it must be clearly understood that this piece of work is not the solution of the Rider, though it often helps you to find the solution. In nearly every case *all* the facts given are necessary.

§ 7. **Spelling.** Naturally several words are unfamiliar. The commonest mistakes are in the words:

Angle,
Base,
Bisect,
Complement (not compliment).
Enunciation (*En* not *Ann*),
Isosceles,
Parallel and Parallelogram,
Rhombus,
Rider (the same as "ride a horse"),
Similarly (it is easy to make the adjective *similar* into an adverb by adding *-ly*).

§ 8. **Style.** The distinction between good and bad wording of the *same* piece of work is illustrated below:

Bad.	**Good.**
(i)	In the Δs ABC, XYZ,
AB = YX, (ii)	$\because$ AB = XY (brief reason),
XZ = AC, (iii)	AC = XZ (brief reason),
$\angle$A = $\angle$YXZ, (iv)	$\angle$A = $\angle$X (brief reason),
etc.	etc.



(i) No introduction.

(ii) No reason given; and XY is better than YX.

(iii) Order reversed, so far details of $\triangle ABC$ were on the left. No reason given.

(iv) There can be no ambiguity here; one letter for the angle is enough. No reason given.

§ 9. **Jumping to conclusions.** By all means jump to conclusions if afterwards you try to pick holes in your reasoning. Things are not always what they seem.

§ 10. **Generalizing from special cases.** Very much *beware of generalizing from a special case.* So in riders about a triangle be careful not to draw an equilateral triangle (or anything near it). In riders about a quadrilateral do not draw a parallelogram (which is only a special quadrilateral), etc.

In drawing a figure take care not to make two things look equal unless there is a reason why they should be.

§ 11. **Look upon a Geometry both as a Guide Book and as a Dictionary.** Using it as the former, you can learn what is important; and you can get practice from the examples. Using it as the latter, you can look up any point you may have forgotten. Into the bargain you should apply your knowledge to every-day affairs, especially out-of-doors; reading the book is useful for giving you ideas, but mere reading is not a substitute for applying those ideas.

§ 12. It must not be assumed that figures are drawn to scale (sometimes, in the book, they are not). It is true that quite often suspicions gathered from the appearance of a *correct* (general) diagram *may* prove to be all right, but **you must rigidly refuse to argue from appearances for a rider** (the fallacies on p. 93, etc. will show to what that may lead) and no measurement is lawful in a rider, for reasoning depending upon measurement is not sound.

TO THE STUDENT.

Throughout, the bookwork and examples are either blank, or have one asterisk (*) at the side, or two asterisks (**) at the side.

It is suggested that in the first reading you should confine your ideas to the unmarked parts. On a second reading you should also tackle the * parts. On a third reading you should be familiar with all.

Last Page.—This reference, in several examples in succeeding chapters, is to the pricking-off figure which appears at the end of the volume.

CHAPTER I.

PRACTICAL GEOMETRY.

(This chapter introduces bisection of lines and angles and perpendiculars. The explanation of the methods of carrying out these constructions will be found in Chapters II. and III.)

§ 1. Before considering any theoretical geometry at all you should become thoroughly accustomed to the following **instruments:**

(i) **Ruler** graduated in **inches** and **tenths of an inch,** and in **centimetres** and **millimetres.** (See page **x.**)

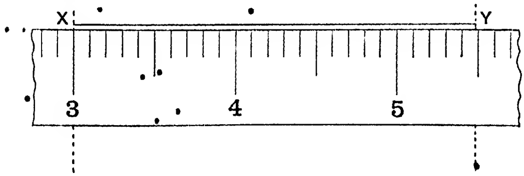
(ii) **Compasses.** (iii) **Protractor.** (iv) **Set square.**

§ 2. The **measurement** of a line XY.

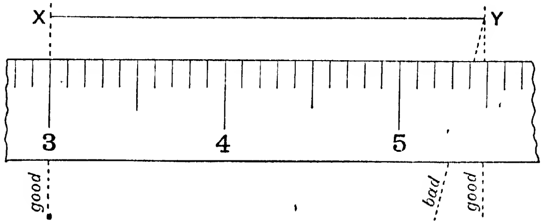
HINTS. (i) The very end of the **Ruler** is apt to become worn, and so you had better avoid the very end.

(ii) Put the ruler along XY. Get a prominent division of the ruler opposite X (don't get your eyes at the side). Hold the ruler tight. Move your head. Get the position of Y by the ruler. See that X is all right. The ruler gives the length 2·4 and a bit. Imagine each tenth of a unit divided up into ten parts and estimate by eye the size of the bit.

Here XY is about 2·48 units.



(iii) The necessity for having the graduations of the ruler quite close to the line XY is best seen by the next figure.



It shows (1) you must have your eyes correctly opposite the ends of the line.

(2) The line must be close to the ruler. (A thick ruler means that the graduations are necessarily far from the line.)

The bad reading is about 2·42 units and the good reading is about 2·48 units.

§ 3. The **Compasses** should be furnished with a hard pencil point.

HINTS. (i) Do not let the pencil in the compasses be so long as to make it difficult to draw a complete circle.

(ii) Hold the compasses at the very top, and not part of the way down one leg. Keep the compasses as upright as possible.

§ 4. With the **Protractor** you can make angles of a given size or measure an angle drawn. The protractors you have are probably graduated in degrees. Sometimes you may be able to estimate, by eye, a fraction or decimal of a degree.

§ 5. A **Set square** can be used for other purposes than for drawing perpendiculars, but first learn to use one for that only. The process is learnt by watching others ; but, in default of that, the system is :

(i) Lay one of the shorter edges of the set square along the line to which a perpendicular is to be drawn, *just hiding the point* from which the perpendicular is to be drawn. Hold the set square tight.

(ii) Put a straight edge (ruler) along the longest side of the set square. Hold the straight edge tight.

(iii) Loose the set square and slide it along the straight edge *until the point just comes in sight.* Hold the set square tight.

(iv) Rule the line.

[*N.B.*—Not sliding the set square generally means a right angle drawn that is not very sharp. Make the line too long rather than too short. Use india-rubber, if you like; but it is really quite unnecessary.]

§ 6. A **separate pencil** for ruling purposes is wisest. It wears out the hinge of the compasses if they are constantly being straightened unnecessarily.

§ 7. A hard pencil is an H (not HB, which is excellent for writing). The lead can be kept sharp with sandpaper.

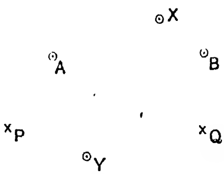
EXAMPLES 1 (GEOMETRICAL DRAWING).

Length.

1. From your scale obtain a length of exactly 5 inches. Measure that length in centimetres. From your measurements calculate the number of cm. in 1 in.

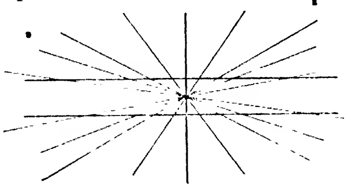
2. From your scale obtain a length of exactly 10 centimetres. Measure that length in inches. From your measurements calculate (i) the number of inches in 1 cm. and (ii) the number of inches in 1 metre.

3. Measure AB, PQ and XY, first all in inches, and secondly all in centimetres.



4. Measure the width of this letter T in cm. and also the height (including the top). Which is the longer?

5. How far apart (the nearest way) are the two lines which do not meet? Measure in several places. Are they wider apart anywhere?



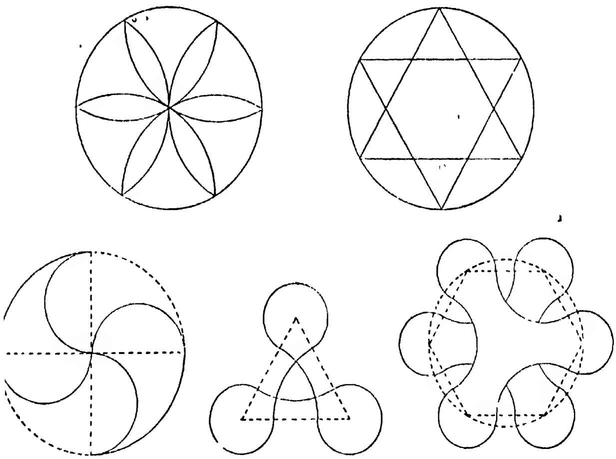
6. Measure in inches the lengths of PR, RS, SP (last page). What would be the total length all the way round the triangle PRS? Notice that it is not necessary to join PR, RS and SP to measure them.

Circles.

7. Two forts are 12,000 yards apart. The guns of one have a range of 9000 yards and of the other 7000 yards. Draw a diagram and shade the area commanded by both guns. [Scale 1000 yds. to a centimetre.]

8. In a circular field, whose radius is 50 yards, there is a stake 10 yards from the centre. A dog is tethered to the stake by a cord 20 yards long. Draw a diagram, and shade in the area over which the dog can roam. [Scale 10 yds. to 1 cm.]

9-13. Draw the accompanying patterns. In the diagrams the radius of the dotted circle (which should be drawn first and very faintly) might be 4 cm. The sides of the equilateral triangle are to be 8·5 cm. and the radii for that figure 6 cm. and 2·5 cm.

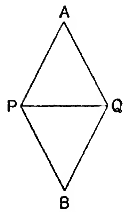


Triangles (with given sides).

14. ABC is a triangular park. BC = 300 yds., CA = 170 yds., AB = 280 yds. A house, H, is situated in the park 200 yds. from

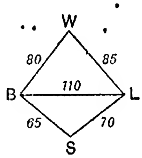
B and 120 yds. from C. Draw a diagram to the scale 100 yds. to the inch. How far is the house from the corner A ?

15. On opposite sides of a line PQ 5 cm. long draw two equilateral triangles APQ, BPQ, each having PQ as base. How far apart are A and B ?



16. PQR is a field. QR = 80 yards, RP = 50 yards, PQ = 70 yards. A treasure is buried in the field. The treasure is not more than 35 yards from P, not more than 55 yards from Q, and not more than 45 yards from R. Draw a correct diagram, and shade in the portion of the field in which the treasure may be.

17. Draw correctly a diagram showing the relative positions of London, Bristol, Southampton and Warwick. The distances in miles are given in the accompanying diagram. How far is it from Warwick to Southampton direct ?



18. A small triangular plot of ground, ABC, is bounded by hedges. BC = 17 yds., CA = 16 yds., AB = 23 yds. There is a stake, S, in the plot, SB = 13 yds., SC = 7 yds. A goat is tethered to the stake by a cord 4 yds. long. Draw a correct diagram, and shade in the portion of the plot that the goat can move over.

Bisection of Lines.

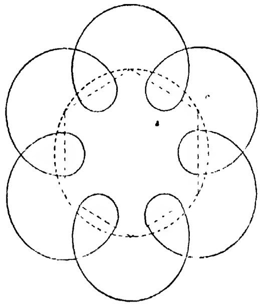
19. Draw a line 2·7″ long. Divide it into 2 equal parts by construction. Measure these parts in millimetres.

20. Draw any (fairly large) triangle ABC. Bisect the two sides AB and AC at X and Y respectively. Join and measure XY in inches. Measure BC in inches. Compare XY and BC.

21. Draw any (fairly large) triangle PQR. Bisect QR at X, RP at Y, and PQ at Z. Join PX, QY and RZ. [They ought to meet at a point if you have drawn correctly.]

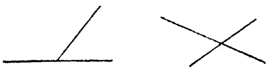
22. Obtain a length of exactly 1·85″ by bisecting a line 3·7″ long. Measure your halves, too, to see how satisfactory they are.

23. Draw the accompanying pattern. First draw (very faintly) the dotted lines.
[Radii 5 cm., 3·8 cm., and 1·3 cm.]

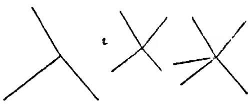


Angles.

24. Draw any two lines meeting as in the accompanying diagrams. Measure each angle, and in it write its size in degrees. Repeat with different figures. Do you suspect any laws which govern their sizes?



25. Draw 3, or more, straight lines meeting at a point. Measure each angle, and in it write its size in degrees. What is the sum total of all the angles in the separate cases? Do you suspect any law?



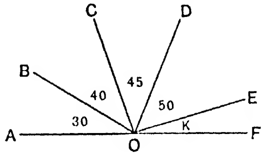
26. Make *any* triangle (fairly large is most convenient). Measure each of its angles, and in it write its size in degrees. What is the sum of all the three angles? Repeat with another (and quite different) triangle. What is the sum of its three angles?

27. Draw *any* four-sided figure (fairly large is most convenient). Measure each of its angles, and in it write its size in degrees. What is the sum of all the four angles? Repeat with another (and quite different) four-sided figure. Do you suspect any law?

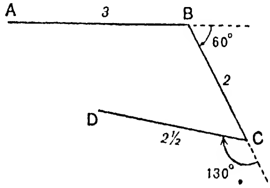
28. Two hands of a clock are 1·2″ and 1·9″ long. The angle between them is 100°. How far apart are their tips?

29. Make a triangle PQR with QR equal to 2·7″ ∠Q equal to 50° and ∠R equal to 66°. What is the size of ∠P ? Measure PQ.

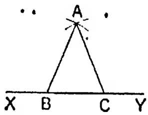
30. AOF is a straight line, and the four straight lines OB, OC, OD and OE issue from some point O on it. The sizes of the angles in degrees are marked on the figure. Draw the figure correctly. Measure the unknown angle K. Do you know what it ought to be ?



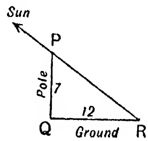
31. Draw correctly a diagram of the accompanying course: AB 3 miles, turn to the right 60°; BC 2 miles, turn to the right 130°; CD $2\frac{1}{2}$ miles. How far is it from D to A direct ?



32. On a base BC 2″ long make a triangle ABC with AB and AC each 2·4″ long. Produce BC both ways to X and Y. Measure the 5 angles, and in each write its size.

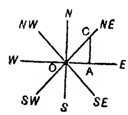


33. An upright pole PQ is 7 feet and it casts a shadow QR 12 feet long. Measure ∠R. At what angle is the sun up ?



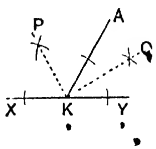
34. With the protractor draw two lines NOS, WOE at right angles to indicate a compass. Then, still with the protractor, get the directions north-east, etc.

A man at O sees a church (C) in the direction N.E. After walking 2 miles due east to A the church is due north. Measure OC.

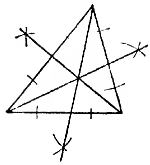


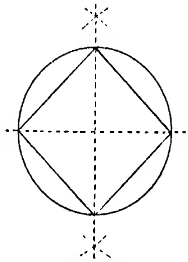
Bisection of Angles.

35. Draw any straight line XKY and any straight line AK meeting it at any point K. Bisect the angles AKX and AKY by PK and QK respectively. Measure the angle PKQ.



36. Draw any (fairly large) triangle. Bisect its 3 angles. [Do the three bisectors meet at a point ?]





Perpendicular Bisectors.

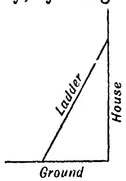
37. Construct accurately a square in a circle, whose radius is 5 cm. Measure its sides.

38. Draw *any* (fairly large) triangle ABC. Bisect the side AB at right angles. Bisect the side AC at right angles. Let these two perpendicular bisectors meet at S. Is the point S nearest to A, B or C ?

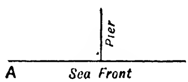
39. Draw a circle with radius $1\frac{1}{2}$ inches. Draw two diameters at right angles. Bisect all four radii, perpendicularly, by straight lines forming a square.

Internal Perpendiculars.

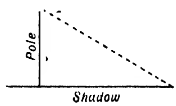
40. The foot of a 21 ft. ladder is 10 ft. from a house. How high up the house does it reach ?



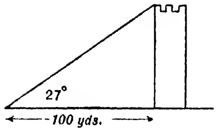
41. A pier, 80 yards long, juts out into the sea at right angles to the sea front. A is a point on the sea front 130 yards away from the pier gates. How far is it from A direct to the pier head ?



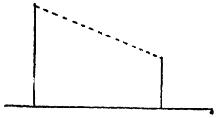
42. An upright pole, 10 ft. high, casts a shadow 20 ft. long. How far is it from the top of the pole to shadow of the top of the pole ?



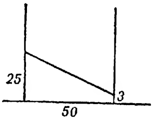
43. A tower rises out of a level plain. 100 yds. away from the foot of the tower the elevation of its top is 27°. How high is the tower?



44. Two upright poles 8 ft. and 4 ft. high are situated on level ground 10 ft. apart. How far apart are their tops?

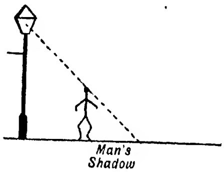


45. A street is 50 feet wide. A window in one house is 3 feet from the ground, and in the house exactly opposite there is a window 25 feet from the ground. How far apart are the windows?



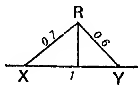
46. A football ground is 130 yards by 90 yards. Draw a diagram of the ground correctly to scale. Measure both of its diagonals.

47. The lamp-post is 8 ft. from the man. The man is 6 ft. high. His shadow is 7 ft. long. How high is the light above the ground?

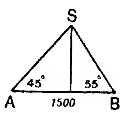


External Perpendiculars.

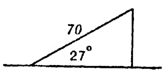
48. X and Y are two houses, on a straight road, 1 mile apart. A railway station, R, is 0·7 mile direct from X and 0·6 mile direct from Y. How far is the railway station to the nearest point on the road?



49. Two coast-guards at A and B, 1500 yards apart, notice a ship (S) at sea. They observe the angles SAB and SBA to be respectively 45° and 55°. How far is the ship from the line AB?



50. A kite is flying with 70 yards of string. The string is straight and makes an angle of 27° with the ground. How high is the kite?



CHAPTER II.

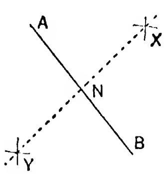
STRAIGHT LINES.

§ **1.** The most obvious example of a straight line is a plumb line, but to be straight a line need not be bolt upright (or **vertical**); a slanting (or **oblique**) line may be straight [for example, the Leaning Tower of Pisa is straight]. In Geometry a line has no breadth, and though the pencil lines we draw are sure to have some thickness, we can aim at getting them fine by using a hard pencil.

§ **2. To draw a triangle with sides of given length.** Draw one of the sides (often it is not a bad plan to take the longest) and, with its ends as centres and with radii equal to the other two lengths, describe arcs to cut, giving the third corner of the triangle. If the triangle has to be drawn on a reduced scale, the scale adopted should be stated.

§ **3.** To **bisect** anything is to divide it into two *equal* parts. (*N.B.*—" Intersect " merely means cut, and " bisect " means cut equally.)

§ **4. To bisect a given straight line** AB. With A and B successively as centres, and with the same (suitable) radius describe pairs of arcs. Let them cut at X and Y. The join XY gives the point N, which bisects AB.

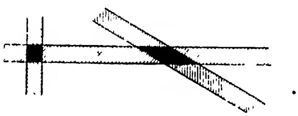


N.B.—(i) For accuracy much depends on " suitable." Aim at getting good intersections. (See § 5.)

(ii) Strictly N is the only point on XY that we need. It will be noted in Chapter III. that the whole line XY is at right angles or perpendicular to AB. The line XY is called the **perpendicular bisector** of AB.

§ 5. **Intersections.**

The illustration of intersections as seen under a microscope affords a reason for the necessity for good intersections in accurate work.



EXAMPLES 2 (GEOMETRICAL DRAWING).

Triangle with given sides.

1. Draw a triangle with sides 4″ and 3″ and 2″. Measure all the sides in centimetres.

2. Draw a triangle with sides 10 cm., 7 cm. and 2·5 cm. [Why is this impossible ?]

3. Draw a triangle with sides 11 cm., 4·9 cm. and 7·6 cm. Measure all its sides in inches.

4. A triangular park has sides 360 yds., 270 yds. and 240 yds. In it a house is situated 190 yds. from each of the ends of the longest side. How far is the house from the corner of the park at which the two shortest sides meet ?

5. On a base BD, 6 cm. long, and on opposite sides of it draw two equilateral triangles ABD and CDB. Measure AC.

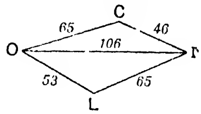
6. On a line QR, 4 cm. long, and on the same side of it draw two triangles PQR and SQR. In the one PQ = PR = 3 cm., and in the other SQ = SR = 6 cm. Measure PS.

7. Draw a kite-shaped figure WXYZ in which WY = 2″, WX = XY = 1·2″, WZ = YZ = 1·6″. Measure XZ. [It is best to

draw a rough figure on which to mark the given dimensions, then it will be seen that it is wisest to begin by drawing WY accurately.]

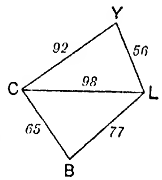
8. The relative positions Oxford, Cambridge, Ipswich and London are shown roughly in the accompanying diagram, in which the distances are given in miles. Draw the figure accurately. Put a title near it. How many miles is it from Cambridge to London direct ?

[Hint. Draw OI first.]



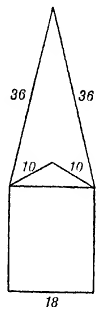
9. Chester to Lincoln = 98 miles.
Chester to York = 92 „
Lincoln to York = 56 „
Chester to Birmingham = 65 „
Birmingham to Lincoln = 77 „

Draw the figure CYLB accurately to scale. How many miles is it from York to Birmingham direct ?

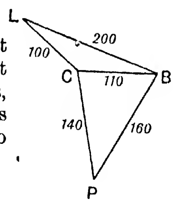


10. A picture is 18″ wide. It is hung by a cord 72″ long from a nail. The cord is shortened to 20″. If the picture is hung from the same nail, how much higher would it be ?

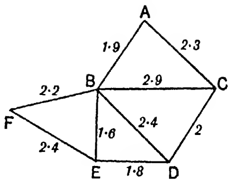
[Hint. In the drawing keep the picture in the same place, and see how much *lower* the nail would be.]



11. The accompanying figure (which is not drawn accurately to scale) gives the direct distances in miles between London, Paris, Brussels and Calais. Draw the positions correctly. How far is it from London to Paris direct ?



12. How far is it from A to F direct? [The accompanying diagram gives distances in miles.]



Bisecting Lines.

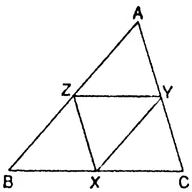
13. Draw a straight line 3·7″ long. Bisect it by construction. Measure the halves in mm.

14. Draw a straight line 7·3 cm. long. Construct the bisector. Measure the halves in inches.

15. Draw a straight line 10·5 cm. long. Divide it into four equal parts by construction. Is each part more or less than an inch long?

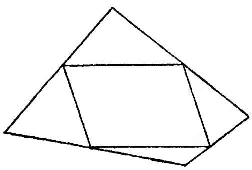
16. Draw a triangle with sides 3·2″, 2·7″ and 2·1″. Bisect the two shorter sides. Join the two points of bisection. Measure the distance between the points of bisection, giving the result in inches.

17. ABC is a triangle with BC 7·8 cm., CA 6·4 cm. and AB 8·2 cm. Bisect each side accurately, by construction, at X, Y and Z respectively. Measure the sides of the triangle XYZ.

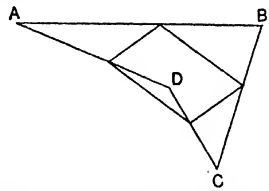


18. ABC is a triangle. BC = 3″, CA = 2·4″, AB = 2·8″. The sides are bisected at X, Y and Z respectively. Join AX, BY and CZ. Measure the longest of those lines.

19. Draw *any* large 4-sided figure. Bisect each of the four sides and join up as indicated. (As a matter of fact do you know the name of the resulting quadrilateral?)



20. Draw *any* large 4-sided figure ABCD, with D re-entrant as in the accompanying figure. Bisect each of the four sides and join up as indicated. [Is the new figure the same shape as the one in the preceding question ?]



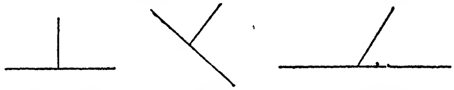
21. Bisect a line 3·7″ long. Make a line half as long again as 3·7″. Measure it in centimetres.

22. Draw a line 10·3 cm. long. Divide it into four equal parts. Make a line a quarter as long again as 10·3 cm. Measure it in inches.

CHAPTER III.

ANGLES.

§ **1. Definition.** When one straight line stands on another straight line so as to make the *adjacent angles equal,* each of those angles is called a **right angle.** [*Adjacent = side by side* or *next.*]

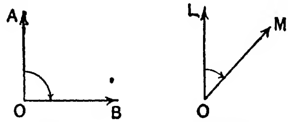


Right angles Right angles Not right angles

§ **2. The size of an angle is independent of the lengths of its arms.** For instance, at 2 o'clock the angle between the hands of a large clock is precisely the same as the angle between the hands of a small watch.

§ **3.** The angle between two directions is **the amount of turning** to get from one direction to another.

Facing in the direction OA the command "Right turn" makes us face in the direction OB. From OL to OM the command is "Half right turn."



Right-turn Half right-turn

A cricket ball is struck to cover-point, and thereby its path has been turned through about $1\frac{1}{2}$ right angles. [Don't think it is only about $\frac{1}{2}$ right angle.]

§ **4.** If two straight lines cross each other they make **four** angles. If those angles are equal, each is a right angle. A right angle is often too big for measuring purposes, and so it is divided up. The four right angles in all are divided up into 360 parts. Each part is called a **degree.** (Because 360 is exactly divisible by 2, 3, 4, 5, 6, 8, 9, 10, 12, and even etc., its convenience is apparent. It may be due to the Babylonians.) That would mean that each right angle is divided into 90 degrees. A degree is a small angle (about an inch in 5 feet), but one which easily can be shown on paper the size of this book. (See page xi.)

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§ **5.** Angles less than right angles are called **acute** angles.

Angles more than one and less than two right angles are **obtuse** angles.

An angle of two right angles is sometimes called a **straight** angle (but the term is not used by practical men).

Between 2 and 4 right angles are **reflex** (or re-entrant) angles. [Latin *reflexus* means bent back.]

Beyond 4 right angles are also considered. [The big hand of a clock would turn through 5 right angles between 3 o'clock and a quarter past 4.]

When two angles together make up 2 right angles (or 180°) they are said to be **supplementary**, or one is said to be the **supplement** of the other. (100° is the supplement of 80°.)

When two angles together make up 1 right angle (or 90°) they are said to be **complementary**, or one is said to be the **complement** of the other. (20° is the complement of 70°.)

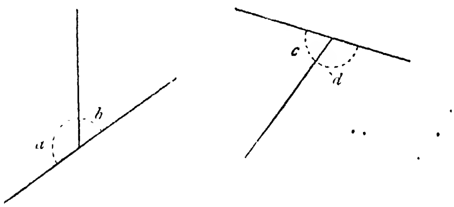
§ **6.** The circumference of a whole circle could be divided up to show degrees, but most often only semicircles are used. Instruments like this are called **Protractors.** With protractors angles can be measured, and can be made of a given size.

§ **7.** For wheels that are rotating fast any particular radius (painted white, say, to distinguish it) would turn through a great number of degrees in a short time, and their speed is reckoned in

Revolutions per minute. [It is easy to change this into degrees recollecting that 1 revolution means 360°.]

§ **8.** A **Traverse** (accent on the first syllable) is the name given to a course, made up of straight lines, whose lengths and directions are known.

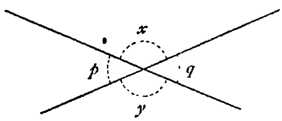
§ **9.** When one straight line falls on (or meets) another straight line, a pair of angles is formed. These angles are called **adjacent** angles. Adjacent angles are next each other or side by side. [Angles are not equal simply because they are adjacent.]



The sum of adjacent angles is 180°. Except in the case when the angles are each 90°, one is acute and one is obtuse. In the figures above b and c are acute, while a and d are obtuse.

Again, *if the sum of some angles is* 180° *the angles can be put side by side, and the first and last arms are in the same straight line.*

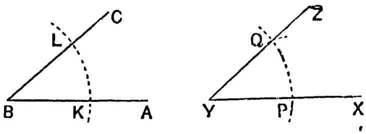
If two straight lines cut one another four angles are formed. Two pairs of equal angles are made. *The* **vertically opposite angles** *are equal.* Here $\angle p = \angle q$ and $\angle x = \angle y$.



§ **10.** An angle can be **copied** as follows :

Suppose that ABC is the given angle and that it is required to

make the same sized angle at the point Y in the straight line XY.



With centre B and any (suitable) radius describe an arc of a circle cutting AB and BC at K and L.

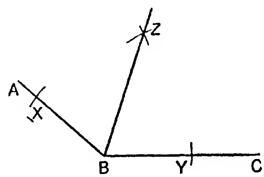
With centre Y and the *same* radius describe an arc of a circle cutting XY at P.

Now open the compasses [a pair of dividers is more convenient] to the distance KL. Transfer the compasses, and with P as centre get Q.

The straight line QY gives the angle XYZ equal to the angle ABC.

§ **11. To bisect a given angle.** ABC is the given angle. With centre B and any suitable radius get the points X and Y. With centres X and Y, and with the same (not necessarily the same as originally) suitable radius get two arcs and let them cut at Z.

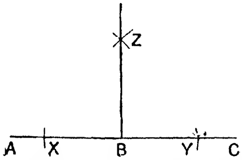
The straight line BZ will **bisect** the angle ABC.



[First take *long* radii (BX, BY); the length will be determined by convenience; lines should not be produced unnecessarily, and compasses should be used well within their limit. Secondly, aim at a "good intersection" at Z.]

§ **12. To construct a perpendicular to a given straight line from a given point IN the same.** AC is the given straight line. B is

the given point IN it. With centre B and with any suitable radius get the points X and Y on AC.



With centres X and Y, and with the same suitable radius (certainly longer than the original) get two arcs and let them cut at Z.

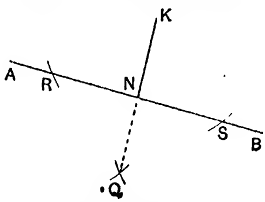
The straight line BZ will be **perpendicular** to AC.

[Aim at getting a " good intersection " at Z.]

N.B.—This construction for a perpendicular is identical with that of bisecting an angle, here 180° exactly, or 2 right angles, and so there is a oneness in sections **11** and **12**.

§ **13. To construct a perpendicular to a given straight line from a given point OUTSIDE the same.** AB is the given straight line. K is the given point OUTSIDE it. With centre K and with any suitable radius get the points R and S on AB. With R and S as centres and with the same (not necessarily the same as originally) suitable radius get two arcs (far preferably on the side of AB away from K), and let them cut at Q.

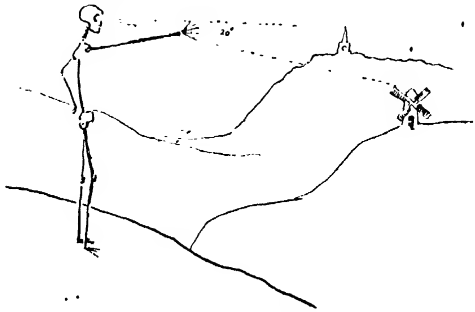
The straight line KQ will be **perpendicular** to AB.



[Aim at getting " good intersections " at R, S and Q. Only the portion KN is usually required.]

§ **14.** The use of a **set-square** is the most convenient way of drawing a perpendicular. [It is wiser to slide the set-square along the ruler than *vice versa.*]

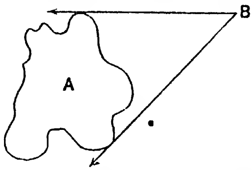
§ 15. **Estimating angles out-of-doors.** Stretch out an arm to full length. Spread out the fingers and thumb as wide as possible. Shut one eye. Two lines, one from the end of the thumb to the open eye and the other from the end of the little finger to the open eye, are inclined to one another at about 20°.



The angle which the line from the spire to the windmill subtends at the eye is about 20°. Fractions of the 20° are estimated, and multiples of the same by "spanning" more than once, starting the second span with the thumb where the little finger rested previously. [Probably about 20° (or less) is appropriate for you; but you should settle the point out-of-doors by seeing how many "spans" you need for the complete circuit of 360°.]

Perhaps the word **subtend** is unfamiliar.

A *thing* subtends an angle *at* a *point*. A is a thing, B is a point.



Imagine yourself at B, looking at the left edge of A. Draw the line along which you would look. Now turn, facing A all the time, until you are looking at its right edge. The angle you have turned through is the angle subtended *by* A *at* B.

EXAMPLES 3 a (GEOMETRICAL DRAWING).

Measurement of Angles.

1. Prick off the points P, R, S (last page). Join up to make the triangle PRS. Measure each of its angles. What is their sum ?

2. Prick off the points P, Q, T (last page). Measure each of the angles of the $\triangle$PQT. What is their sum ?

3. Draw *any* large triangle. Measure its three angles. What is their sum ?

4. Prick off the points P, Q, R, S, T (last page). Join them up to get the five-sided figure PQRST (producing the sides well for convenience of measurement). Measure the 5 angles. What is their sum ?

5. With your protractor measure the six angles A, B, C, etc., on the last page. Give their sizes in degrees. [*N.B.*—The angle C is reflex.] What is the sum total of all six angles ?

Traverses.

6. A straight road starts from a village, and after mounting for a mile at an average rise of 8°, descends for half a mile at 4°. Draw a diagram to illustrate this. At what angle would a road go direct from start to finish ?

7. A man starts at K and walks 3 miles to L; he then turns to the right through an angle of 45° and walks $2\frac{1}{2}$ miles to M; he then walks back in a straight line to K. How long was his walk altogether ?

8. A man walks northwards from A to B, 2 miles; he then turns to the right through an angle of 30° and walks 2 miles further to C, then turns to the left through an angle of 60° and walks 3 miles to D. Measure the distance in a straight line from A to D.

9. A surveyor in mapping a new region travels 5 miles in a straight line, turns to the right 53°, proceeds 4 miles, again turns to the right 37° and proceeds 3 miles. How far is he then from his starting point ?

Copying Angles.

10. Copy the angle D (last page) by construction. Measure the original and the copy. Do they agree ?

11. Copy the angle P (last page) by construction. Measure the original and the copy. Do they agree ?

12. Draw a triangle ABC with sides BC 4″, CA 3″ and AB 2″. By copying the angles B and C only, construct a triangle XYZ similar in shape to ABC. [Y corresponds to B and Z corresponds to C.] Make YZ 12 cm. Measure the sides ZX and XY in cm.

13. With your protractor make an angle of 17°. By construction double it. Measure the resulting angle. Is it satisfactory?

Bisecting Angles.

14. Prick off the points P, T, R (last page). Construct the bisector of $\angle$PTR. Measure the $\angle$PTR and its halves.

15. Construct a triangle ABC with $\angle C = 79°$, $CA = 4{\cdot}7$ cm., $BC = 6{\cdot}1$ cm. Measure the remaining parts of the triangle and bisect the smallest angle.

16. Draw a triangle ABC, in which $AB = 3{\cdot}6''$, $BC = 2{\cdot}4''$, $\angle ABC = 112°$. Bisect the angles A and C, and let the bisectors meet at O. Join OB; and measure OB and $\angle$OBC.

17. Prick off the points A, C, E (last page). Divide the angle ACE into 4 equal parts by construction. Measure each part.

18. With the protractor make an angle of 35°. Bisect it. Make an angle half as large again as the original. Measure it.

Internal Perpendiculars.

19. Draw any straight line XY. Take any point P on it From P construct a perpendicular to XY. Measure the two angles that the perpendicular makes with XY.

20. Prick off the points P, Q (last page). On your paper rule in the straight line PQ, prolonging it well beyond P. From P construct a perpendicular to PQ. Measure the two angles.

21. Draw a straight line ABC 4·5″ long, such that $AB = 2{\cdot}5''$ and $BC = 2''$. From B construct a perpendicular BP (2″ long) to ABC. Measure AP and CP.

22. Two upright poles, 5 feet and 8 feet high, are erected on level ground. Their feet are 10 feet apart. Construct a diagram showing this (scale 1 cm. to 1 foot). How far apart are their tops?

External Perpendiculars.

23. Prick off the points P, T, Q (last page). Then rule in the straight line TQ. From P construct a perpendicular to TQ. Measure in cm. the distance from T to foot of the perpendicular.

24. Prick off the points A, P, F (last page). Construct the perpendicular from A to PF. Measure the distance in cm. between P and the foot of the perpendicular.

25. Draw a triangle ABC in which AB = 3·6 cm., BC = 5·8 cm., $\angle$ABC = 108°. From B construct BD perpendicular to AC. Measure AD in cm.

26. ABC is a triangle in which AB = 10 cm., $\angle$A = 27°, $\angle$B = 90°. D is the foot of the perpendicular from B on AC. Measure the lengths of BD and DC.

27. A 50-foot ladder, AB, leans up against a house BC, making an angle of 70° with the horizontal line AC. Find how high up the house the ladder reaches.

A man, M, climbs 35 feet of the ladder: what is then his vertical height from the ground, and his horizontal distance from the wall? Find also the size of the angle MCA, and the length of MC.

Perpendiculars with a set-square.

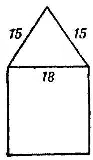
28. Prick off the points P and R (last page). Join PR. With the set-square make perpendiculars to PR, each 2″ long and on opposite sides of PR. Mark them PF and RG. Join FG. As a matter of fact, how does it treat PR?

29. Draw a straight line AB 3″ long. At the points A and B, and on the same side of AB, make perpendiculars AD and BC, each 2″ long, with a set-square. Join CD. As a matter of fact, do you know the name of the shape ABCD?

30. Prick off the points A, F, Q (last page). With a set-square draw a perpendicular from F to AQ.

31. Draw a triangle PQR with PQ 10 cm., QR 8 cm., and the contained angle PQR 40°, and, with a set-square, from P draw PX perpendicular to QR. Measure QX.

32. A picture 18″ wide is hung from a nail by 30″ (15″ + 15″) of string. How much must the string be lengthened to lower the picture 1 ft. if it is hung from the same nail?



Miscellaneous.

***33.** Draw any straight line ABC. At the point B in it construct a perpendicular to it. Divide one of the right angles into 4 equal parts by construction.

* **34.** Draw a triangle ABC on a base BC = 8 cm., so that angle B = 80°, C = 35°; bisect the angle A by a line AD, and construct a perpendicular AX from A on to BC. Measure angle DAX.

* **35.** Draw a triangle whose base angles are 23° and 67° and whose base is 6 cm. Measure the height and other sides.

* **36.** ABC, ACD, DCE form a chain of three triangles (each nearly equilateral). AB = 7 miles, and the following angles are observed:

CAB = 57°,	CBA = 62°,
DAC = 60°,	DCA = 58°,
EDC = 59°,	ECD = 61°.

Draw the figure to scale and measure DE.

* **37.** A subway ABCD under a river first descends at 15° for 200 feet from A to B, then descends at 8° for 400 feet from B to C, and lastly ascends at 12° from C to D. What is the distance CD (assuming that A and D are at the same level)?

What length of the subway is less than 40 feet below the level AD?

* **38.** Draw a triangle ABC, given AB = 6 cm., BC = 7 cm., AC = 8 cm. Bisect the angles at B and C by lines meeting at P. From P draw PM perpendicular to BC, and with centre P and radius PM describe a circle. Measure PM in millimetres.

* **39.** A subway descends for 500 feet at an angle of 8°, is level for 200 feet, and ascends for 300 feet at an angle of 12°. What is the difference in level between its two ends?

The subway goes under a canal: what length of the subway would be flooded by water from the canal, if the surface of the canal is 40 feet above the lowest part of the subway?

* **40.** ABC is a straight piece of the seashore (AB is 100 yds. long and BC is 150 yds.). At B a pier juts out into the sea perpendicularly from the shore. The pier is 70 yds. long. Construct a diagram to scale, and measure the distance direct from the point A to the pierhead.

* **41.** Draw a triangle ABC with base BC 4·62″ and base angles 30° and 60°. Construct and measure the altitude AD.

* **42.** A man walks 100 yds., then turns to the right through an angle of 72°. He repeats this 4 times (5 in all). Draw a figure (100 yds. to an inch) and show how far he is from the starting point when he has finished.

* **43.** A man starts walking at A, and goes in a straight line to B, a distance of 300 yards. He then turns to the right through

an angle of 50°, and walks 250 yards to C. At C he again turns to the right through an angle of 70°, and walks 100 yards to D, and at D he again turns to the right through an angle of 60° and walks 170 yards to E. Draw a plan of his walk on any convenient scale (preferably 100 yards to the inch) and measure the length in yards of the straight line AE.

* **44.** To get to the other side of a steep hill I have first to drive 1200 yards along a straight road; then turn to the right through an angle 75°, and drive 500 yards; lastly to the right at right angles for $\frac{3}{4}$ of a mile. Draw a plan of the drive, on a scale of 4 inches to a mile, and find how far I am at the end from the starting point.

* **45.** A man observes the angle of elevation of the top of a hill to be 13°. He walks a mile towards it and then finds the angle to be 30°. Find the height of the hill.

* **46.** The height of a building is to be determined from the following observations:

At station A, elevation of top of building 44°.
At station B, elevation of top of building 64°.

AB is 115 ft., also A and B are in the same straight line as the foot of the building, and at the same level. At A and B the instrument, by which the elevations were obtained, was 5 feet above the level ground. Find the height of the building above the ground.

* * **47.** To two points on exactly opposite sides of a tower the angles of depression are 30° and 48° from the top of the tower, and the distance between the points is 50 yards. What is the height of the tower?

* * **48.** From the masthead of a ship the angles of depression of two buoys lying in the same direction are 10° and 20°. If the buoys are 30 yards apart, what is the height of the masthead?

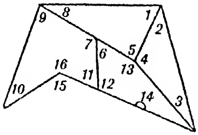
* * **49.** A picture 2 ft. high hangs by two parallel cords each 4 ft. long. If the picture makes an angle of 15° with the wall, how far is the top edge from the wall, and what angle do the cords make with the wall? [The cords are fixed to the top edge of the picture.]

* * **50.** A cubical block of edge 4 feet rests on a table; the base is ABCD, and P, Q, R, S are the corners above A, B, C, D respectively. If the edge is raised 2 feet, AB remaining on the table, find, by drawing to scale, the height of R above the table, and the inclination of AR to the table.

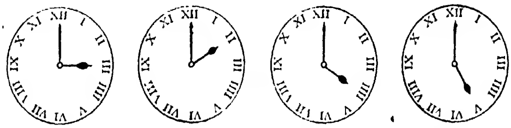
EXAMPLES 3 b (CALCULATIONS).

Some questions might be discussed orally. ("Answers" to the first 11 *questions are not given.)*

1. Make a list of the numbers 1, 2, 3, ..., 15, 16. From appearances only (no measurement is wanted), say whether each of the angles is acute, right, obtuse, straight or reflex.



2. How many degrees are there in the angles made by the hands of a clock in the four following cases ?



3. The wind changes from south to west; through how many degrees does the weathercock turn ?

4. How many degrees do the commands (i) "Left turn" and (ii) "Half-left turn" imply ?

5. How many degrees does the command "About turn" imply ?

6. What are the supplements of the following angles: 60°, 100°, 135°, 90° and 57° ?

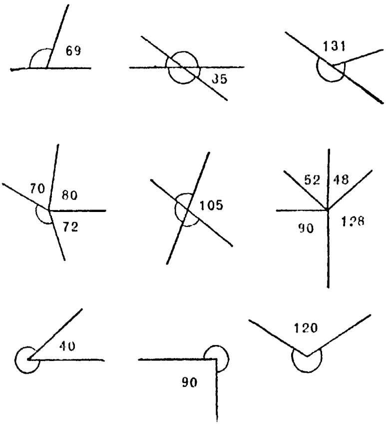
7. What are the complements of the following angles: 80°, 30°, 2°, 43°, 45° ?

8. Of course at 6 o'clock the hands of a clock would be just opposite. How many degrees would there be in the angle they formed ?

9. How many degrees has the little hand of a clock moved from the preceding noon at 4 p.m., 5 p.m., 6 p.m. and 7 p.m. ?

10. Answer the same question, but the times are 4 a.m., 5 a.m., 6 a.m. and 7 a.m. (*N.B.*—From preceding noon, not midnight.)

11. In the following cases calculate the unknown angles. [The sizes of those given are in degrees.]



12. If the screw of a ship is revolving 90 times a minute in the water, how many degrees is that per second ?

13. A shell in the barrel of a gun revolves (owing to rifling) $1\frac{1}{2}$ times. How many degrees is that ?

14. How many degrees are turned through, in 18 minutes of time, (1) by the hour hand, (2) by the minute hand, and (3) by the second hand of a watch ?

15. A bicycle is going along a road at 15 miles per hour. The circumference of one of its wheels is 88″. How many revolutions is it making a minute ? How many degrees does one of its spokes turn through in a second ?

16. Owing to gearing, of course the pedals do not turn so fast as the wheels of a bicycle. Suppose that they are revolving 3 times as slowly ; through how many degrees do the pedals turn in 1 second when the bicycle is travelling 20 miles an hour ? [The circumference of the wheels is 88″.]

17. An aeroplane engine is making 1300 revolutions per minute. How many degrees is that per second ?

18. A locomotive engine is going at 60 miles an hour. If the circumference of its driving wheels is 22 ft., how many

revolutions are they making a minute? Through how many degrees does one of its spokes turn in a second?

19. Taking the circumference of a carriage wheel to be 10 ft., how many revolutions does it make per mile? If one particular spoke is painted white (to distinguish it), through how many degrees may it be seen to turn when the carriage has gone forward 3 ft.?

* **20.** In a bicycle a cogwheel having t teeth is attached to the hub of the driving-wheel, which is of diameter d inches. An endless chain passes round this cogwheel and round another cogwheel to which the pedals are attached by means of cranks, the latter cogwheel having T teeth. Find how many revolutions are made by the cranks per minute when the bicycle is travelling at v miles per hour.

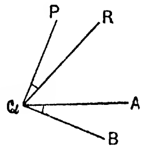
Give the numerical result when $t=16$, $T=50$, $d=28$, $v=14$.

EXAMPLES 3 c (RIDERS).

1. On the arms of the angle AQR (and outside it) are two equal angles PQR and AQB.

Prove $\angle PQA = \angle RQB$.

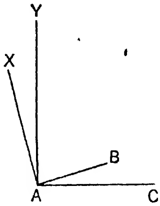
[HINT. $\angle PQR = \angle AQB$ (why?); to each add $\angle RQA$, etc.]



2. BAC is any angle. XA is at right angles to AB (so that $\angle XAB =$ rt. $\angle$), and YA is at right angles to AC (so that $\angle YAC =$ rt. $\angle$).

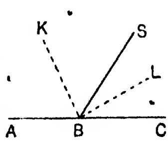
Prove $\angle XAY = \angle BAC$.

[HINT. First $\angle XAB = \angle YAC$. Why?
From each subtract $\angle YAB$.
Conclusion what?]

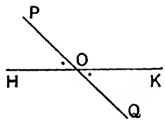


3. SB is any straight line meeting the straight line ABC. KB is the bisector of the angle ABS, and LB is the bisector of the angle SBC. Prove that the angle KBL must be a right angle.

[HINT. What fraction of $\angle ABS$ is $\angle KBS$? What fraction of $\angle CBS$ is $\angle LBS$?
Add the results.]

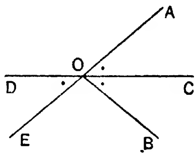


4. If from the point O in the straight line HK lines OP, OQ be drawn on opposite sides of HK so that the angles HOP, KOQ are equal, show that POQ is a straight line.



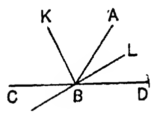
[HINT. $\angle$HOP = $\angle$KOQ (why ?); to each add $\angle$HOQ; $\therefore$ etc.]

5. The angle AOB is bisected by a straight line OC; if CO be produced to D, and also AO be produced to E, prove that the angles BOC, DOE are equal.



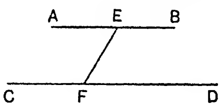
[HINT. $\angle$AOC = $\angle$BOC (why ?).
Again, $\angle$AOC = $\angle$DOE (why ?).
Conclusion ?]

6. The straight line AB meets the straight line CBD at B. BK is the bisector of the angle ABC, and LB is perpendicular to BK. Show that LB bisects the angle ABD.

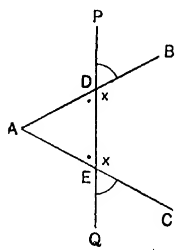


[*N.B.*—This question (though like Question 3) is different from it.]

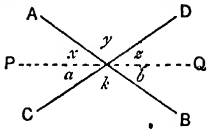
* 7. E, F are two points in the straight lines AB, CD respectively, and EF is joined; show that if the angles AEF, CFE are supplementary, so also are the angles BEF, EFD. Prove also that if the angles BEF, EFC are equal, the angles AEF, DFE must also be equal.



* 8. Two straight lines AB, AC are cut by another straight line PQ at D, E respectively, so that the angles ADE, AED are equal; prove that (1) the angles BDE, CED are equal, (2) the angles PDB, QEC are equal.



*9. If two straight lines intersect, prove that the bisectors of each pair of vertically opposite angles form one straight line.



[HINT. $a+x=b+z$ (why ?),
and $a=x$ (why ?),
and $b=z$ (why ?).
Conclusion $a=z$ (why ?);
but $a+x+y=2$ rt. $\angle$s (why ?);
$\therefore z+x+y=?$
Conclusion what ?]

CHAPTER IV.

ANGLES (continued).

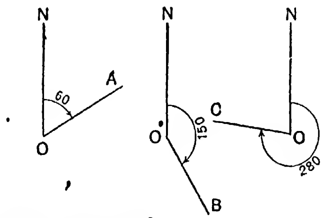
(*This chapter should be omitted entirely in a first reading.*)

* § 1. An angle of one degree (or a half or third of a degree) is small enough for most drawing, but is not small enough for accurate surveying. It is divided up into **minutes** (*partes minutae*) and **seconds** (*partes minutae secundae*).

1 right angle	= 90 degrees	(90°),
1 degree	= 60 minutes	($60'$),
1 minute	= 60 seconds	($60''$).

[A degree is about an inch in 5 feet.
A minute is about an inch in 100 yards.
A second is about an inch in 3 miles.]

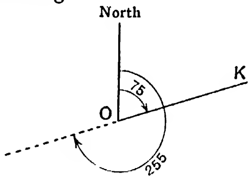
* § 2. For **bearings** a standard direction is required. The standard direction is North. Bearings are reckoned clockwise.



The bearing of OA is 60°.
" " OB is 150°.
" " OC is 280°.

* § 3. **Back-Bearings.** On a *flat* surface the back-bearing differs from the forward bearing by 180°. [For small areas, say the size of England, the curvature of the Earth is negligible in G.D.]

In the figure following, if OK is the forward bearing, then KO is the back-bearing.

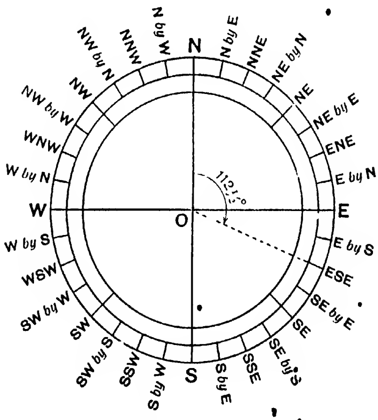


The bearing of OK is 75°.

" " KO is 255°. $(180+75=255.)$

[If you get more than 360 by addition it is usual to subtract 360 at the end; one talks about a bearing of 10° rather than about a bearing of 370°.]

* § 4. At sea, and often on land too, a **Mariner's Compass** is used. The 32 " points " are indicated in the diagram. Every



" point " is $11\frac{1}{4}°$, for $\frac{360}{32}=11\frac{1}{4}$. [It must be recollected that angles at the centre are implied by the arcs, and the direction

E.S.E. means a bearing of $112\frac{1}{2}°$, for the angle N – O – E.S.E. $=112\frac{1}{2}°$.] It is possible to give a direction in many ways, for instance,

Bearing $11\frac{1}{4}°$, or N. by E., or N. $11\frac{1}{4}°$ E. are all the same.

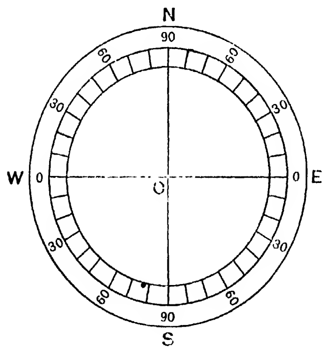
* § 5. The position of a point on any globe may be defined by **its latitude and longitude.**

First, do not attempt to understand the system merely by reading a book and not, at the same time, having a globe handy (essentially a *globe* and not a *flat* map out of an Atlas).

Secondly, find Mecca on the globe (or see the figure on page 39). Call it M. Then on the Equator call the point exactly south of M by the name K. If O is the centre of the globe (of course O is inside the globe and not visible in the figure), then $\angle MOK = 21°$, or Mecca is in north latitude 21°.

Similarly, on the Equator exactly north of Cape Town (C) is a spot near the Congo River (R). Then $\angle ROC = 34°$; or Cape Town is in south latitude 34°.

For the angle of **latitude** the essential thing is to reckon from a spot on the Equator, N. or S. as the case may be. We

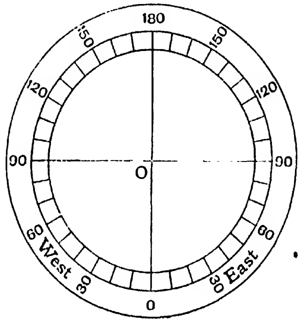


can imagine the globe cut into two hemispheres through the N. and S. poles and the angles marked at the centre. The figure represents this section. NOS is the polar axis, WOE

an equatorial diameter. The right angle NOE is divided up into degrees, and these are numbered 0° to 90° from Equator to Pole. The section could be in any direction through the poles, so could be imagined spun round the axis NOS, giving the **parallels of latitude.**

[So that parallels of latitude are lines (circular on a perfect globe) in planes parallel to the Equator.]

* § **6.** Thirdly, another section, perpendicular to the former, through the Equator is taken. An arbitrary radius, from which to reckon, is chosen; (there is no obvious place to start



from like the Equator for latitude). Angles are marked both ways from that radius. Sections through the N. and S. poles are imagined cutting the equatorial section in order, giving **meridians of longitude.**

[Meridians of longitude are essentially not parallel. They meet at the poles and are furthest apart at the Equator.]

Both E. and W. longitudes are reckoned. In Great Britain, and in many other places, longitudes are reckoned from the meridian of Greenwich.

Now consult the globe again (or see the figure on page 39). X is the point at which the Meridian of Greenwich meets the Equator, then $\angle XOK = 40°$, or Mecca has an East longitude 40°.

The position of Mecca, both latitude and longitude, is summed up in the next section.

* § 7. In this globe the **parallels of latitude** and the **meridians of longitude** are shown at every 15°. (As a matter of fact, in the figure the North latitude line 75° is just visible, but the South latitude line 75° is invisible.)

Mecca is at M.

K is the point on the Equator *nearest* to M, *i.e.* on the M meridian.

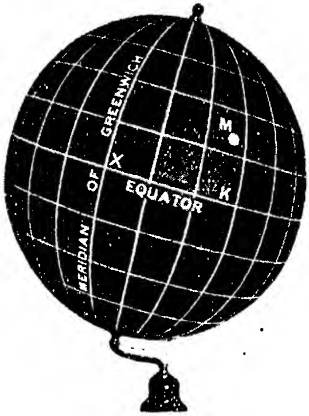
X is the point at which the Greenwich meridian meets the Equator.

If O is the centre (not shown, of course, in the figure) of the globe,

$\angle$MOK = lat. of M = about 21°,

$\angle$KOX = long. of M = about 40°.

The position of Mecca is about latitude 21° N., longitude 40° E.



* § 8. The Earth has a circumference of about **25,000 miles**.

1° of latitude is about $\frac{1}{360}$ of 25,000 miles = 70 miles (about). The Earth revolves on its axis once (360°) in 24 hours, *i.e.* 15° an hour. Thus, if Bombay is in 75° E. longitude, it follows that its time differs by 5 hours [$\frac{75}{15}=5$] from the time at Greenwich, and since Bombay is east of Greenwich (and so sees the Sun earlier), its time is later than that at Greenwich [Greenwich noon, Bombay 5 p.m.].

So that we can get **distances** for differences of **latitude** for two places on the same meridian. (The distance between 2 places *on the Equator* is a very like problem, for the Earth is so nearly round.) We can get **time** for differences of **longitude** (whether the latitudes are the same or not).

[It might be noted that 1 second of latitude, on the Earth, is about 100 ft., so that 2 spots, exactly N. and S. of each other, and within a fine throw of a cricket ball, would have latitudes differing by 3″.]

[The question as to the distance between two places both of different latitudes and longitudes is not treated in an elementary text-book.]

* § **9.** On the figure of the globe (page 39) the meridians are shown at every 15°; and (since $\frac{360}{15}=24$) that means they are shown at hourly intervals. The **Time** at Mecca is about $2\frac{2}{3}$ hours ahead of Greenwich (Greenwich 5 p.m. means Mecca 7.40 p.m.).

* § **10. Standard Meridian for Time throughout a country.** To avoid the hopeless inconvenience of a constant change of time for trivial changes of longitude, the time of a special meridian is taken for the legal time for a wide range. In Great Britain the time of Greenwich (the principal observatory) is taken, and on this account alone there is 23 minutes, or so, difference between actual local time and legal time in the west of Cornwall (this is quite apart from Summer Time).

* § **11.** For countries which have a wide extent of longitude the discrepancy would be too great. Canada adopts **Zone Time,** and, to cover a range of many hours, has Atlantic Time, Eastern Time, Central Time, Mountain Time and Pacific Time, each differing from the next by 1 hour or 15° of longitude. (Thus, if it is noon by Atlantic Time, it is 8 a.m. by Pacific Time.)

The meridians on which the times are actually correct are "midway," so that the legal times may be $\frac{1}{2}$ an hour fast or slow of true local time. If a meridian, where a change should occur, happened to pass exactly through a town or village, of course *one* definite time is chosen for that area to obviate the inconvenience of *two* different times.

Time-keeping at sea. To avoid the great inconvenience of every ship keeping her own time, the navies of Great Britain, France and Italy have adopted a system of Time Zones. You are strongly advised to get the Admiralty Map.* A study of its explanations and examples is most instructive.

* *The World, Time Zone Chart.* J. D. Potter, Agent, 145, Minories, London, E. 1.

* § 12. By travelling eastward we should gain 12 hours in 180°, and by travelling westwards we should lose 12 hours in 180°; and so on the meridian exactly opposite to that of Greenwich there is a discrepancy of a full day (24 hours). Now this meridian is almost entirely in the Pacific Ocean; yet it does pass through some groups of islands; and the inconvenience of having different dates in quite neighbouring places has led to the adoption of the **Date Line.** It does not quite follow the 180° meridian.* On the Asiatic side the date is 1 day ahead of that on the American side (*e.g.* Asiatic side just before 3 p.m. on Wednesday, 12th November, and on American side just after 3 p.m. on Tuesday, 11th November).

* * § 13. **Representing spherical surfaces flat.** Now it is quite impossible to show on a *flat* surface (*e.g.* an atlas map) **correctly** a portion of the surface of a *globe*, and so the system on which the parallels of latitude and meridians of longitude are drawn on a map never gives a perfect representation. Any definite system is called a Projection; all sorts of features—area, direction, length—are important in different circumstances, and it is a consideration of the use to which the map may be put which will form a guide to the choice of projection (*e.g.* area is everything for political purposes, direction is everything for navigation). Say what you think is the shortest way, on the surface of the Earth, direct from London to Sydney (Australia). Then stretch a piece of string *on a globe* tightly between those places, and it may surprise you that it goes close to Petrograd. Of course these discrepancies are very slight in maps of a small area, and England, by itself, on flat maps, can be nearly perfect, but beware of taking too much for granted on a flat map which shows a whole sphere or hemisphere.

[On Mercator's Projection Greenland looks as big as South America, whereas really the area of Greenland is about $\frac{1}{10}$ that of South America.]

* Again see the map, *The World, Time Zone Chart.*

EXAMPLES 4 a (GEOMETRICAL DRAWING).

Compass and Bearings.

*1. A is a lighthouse. B and C are two ships 7 miles apart. B is due south of A, C due west of B, and C south-west of A. Find the distance of each ship from the lighthouse.

*2. A is a lighthouse, B and C are two ships 3·5 miles apart. B is due north of A, C due east of B, and C north-east of A. Draw a plan on the scale of an inch to the mile, and find the distance of each ship from the lighthouse.

*3. A is 12 miles north-east of C, B is 5 miles E.S.E. of C. Measure the distance of A from B, and the bearing of AB.

*4. A is 2·5 miles west of B, and C is 4·5 miles south of A. Draw a plan (scale 1 inch to 1 mile) and find the distance from B to C. Also find the shortest distance from A to BC.

*5. A is N. of B and E. of C. C is 10 miles N. 55° W. from B. Measure AC and AB.

*6. A man walking due E. along a straight road finds that a church spire bears N. 60° E. A mile farther it bears N. 30° E. How far is the church N. of the road ?

*7. A point of land bore from a ship N. 35° W. After she had sailed S. 55° W. 20 miles it bore N.E. What was her distance from it at each observation ?

*8. Two aeroplanes start at the same time from the same field, one going N.W. at 50 miles per hour, and the other N.N.W. at 42 miles per hour. Find the rate at which the distance between them is increasing.

*9. Prick off the points P, Q, R, S, T and the extremities of the line OY (last page). If OY is a north line, measure the bearings of the five lines PQ, QR, etc.

*10. Prick off the points A, B, C, D, E, F and the extremities of the line OY (last page). If OY is a north line, measure the bearings of the six sides of ABCDEF.

*11. A ship is steaming at the rate of 12 knots N. 14° E. past a coast. On a promontory is a lighthouse bearing N. 16° W. In 5 minutes time the lighthouse bears N. 46° W. Fix the

position of the ship and measure her distances from the lighthouse at both observations. Scale 2″ to 1 nautical mile.

[12 knots = a speed of 12 nautical miles an hour.]

*12. O and P are points on a straight stretch of shore, P is 4·5 miles N. 74° E. of O. From O a ship at sea bears S. 58° E., and from P the ship bears S. 32° W. Draw a figure to scale (1 mile = 1″), and find the distance of the ship from P, and also her distance from the nearest point of shore.

*13. A and B are points on a straight stretch of shore 2·5 miles apart. The bearing of B from A is N. 82° E. From A a ship at sea bears S. 43° E. and from B the same ship bears S. 22° W. Draw a figure to scale and find the distance of the ship (1) from A, (2) from B, and (3) from the shore.

*14. A ship steaming N. 50° W. 20 knots observes a lighthouse bearing N. 35° W. at noon and N. 20° W. at 12.30 p.m. Find

(1) her distance from the lighthouse at 12.30 p.m.;
(2) how near she will pass to the lighthouse.

Back-Bearings.

*15. Prick off the points E, B and the extremities of the line OY (last page). If OY is a north line, measure the bearing of the line EB. What is its "back-bearing" (*i.e.* the bearing of the line BE)?

*16. Prick off the points F, B, S and the extremities of the line OY (last page). If OY is a north line, find the position of a point K, such that from K, B bears 40° and F bears 330°. Measure KS in centimetres.

*17. Prick off the points T, R, A and the extremities of the line OY (last page). If OY is a north line, find the position of a point L, such that from L, R bears 140° and T bears 225°. Measure LA in inches.

*18. Supposing the last page represents a map on the scale 1 inch to a mile and that OY is a north line. I wish to identify my position. P bears 340° and E bears 240°. How far am I from Q?

*19. If the last page represents a map (OY being a north line) on a scale 1 : 40,000—I have the map with me. It is a strange

country to me and, of course, I say, "Where am I ?"—the bearing of a church spire (C) is 75° and of a tower (T) is 10°. How far am I from the village Q ?

* **20.** If the last page represents a map (OY being a north line) on a scale 1 : 10,000, and I wish to identify my position, the points D and F respectively bear 75° and 350°. Prick off the appropriate points. Mark my position with the letter Z. How many yards am I from T ?

* **21.** From a point P on the shore a lighthouse is observed to bear S. 26° E. What is the bearing of P from the lighthouse ?

* **22.** From a ship at sea, P is seen to be S. 71° W. and a lighthouse bears S. 12° W. What is the bearing of the ship from each ? If P is 11 miles from the lighthouse and also from S, find (by drawing) the distance of the ship from the lighthouse.

* **22.** A and B are two landmarks 4 miles apart B being N. 65° E. of A. From a ship at sea A bears N. 35° W. and B bears N. 16° E. Draw a figure to scale, and find by measurement the distances of the ship from A and B.

* **24.** A, B, C are three places, such that from A the bearing of C is N. 10° W. and the bearing of B is N. 50° E. ; whilst from B the bearing of C is N. 40° W. If the distance between B and A is 10 miles, find the distances of B and A from C.

Traverses.

* **25.** From A to B is 3 miles N.W. and from B to C is 4 miles N.E. Find the distance from A to C direct and the bearing of the line AC.

* **26.** The orders for a night-march are "Proceed on a bearing of 55° for 1200 yards, then on a bearing of 125° for 800 yards." What is the direction of the objective in a direct line from the start ?

* **27.** A ship sails 4 miles on a bearing of 29°, then 5 miles on a bearing of 136°, and then straight back to the starting-place. How long, and on what bearing, is the last piece sailed ?

* **28.** ABCD is a road, all in a vertical plane, and the following lengths and gradients are known :

AB	300 yds.	up 3°
BC	200 yds.	level
CD	250 yds.	down 2°

(i) How high is D above A ?
(ii) How far is it from D to A horizontally ?
(iii) If another road went from A to D direct at a constant slope, what would that slope be ?

* **29.** An airman flies 10 miles due north, then 20 miles due west, then 5 miles north-west, then 10 miles due north. Draw a plan of his course. If he flies straight home, in what direction must he fly, and how far will he travel ?

* **30.** The orders for a night-march are 3000 yards W. 23° S., then 2000 yards W. 57° S., and then 1000 yards S. 15° E. Find the direction of the point of attack in a straight line from the starting-place.

* **31.** A yacht sails 3000 yards N. 55° E., then 1000 yards N. 32° W., and lastly 2000 yards N. 50° E. How far is she in a direct line from her starting-point ?

* **32.** A survey is run 3·47 chains 18° north of east from A to B, 4·50 chains due east from B to C, and 3·16 chains 70° north of east from C to D. Find how far north and east D is of A.

* **33.** Find the distance from A to D direct and the bearing of the line AD from the following details of a traverse :

Lap.	Distance.	Bearing.
	Yards.	Degrees.
AB	317	63
BC	435	157
CD	567	270

* **34.** Find, by drawing, the distance from A to D direct, and the bearing of the line AD.

Lap.	Distance in Yards.	Bearing in Degrees.
AB	79	48
BC	135	126
CD	91	286

* **35.** A man walks from A to B 4·3 miles N. 20° E., from B to C 3·6 miles E. 15° N., from C to D 3·66 miles S. 35° W. Draw a plan of the walk to a scale of 2 miles to the inch, and measure AC and AD.

* **36.** A ship starting from A sails 20 miles N.E. to B, then 35 miles E. 30° S. to C, then 30 miles W. 15° S. to D.

Draw a plan to scale, measure the distance AD, and also give the bearing of A from D.

* **37.** A ship is sailing due east. A man on board observes two landmarks A and B. At first A is found to be N. 30° E., while B is N. 60° E. After proceeding a mile B is due north, while A is N. 30° W.

Find the distance between A and B, and how B lies with regard to A.

* **38.** A ship sails N.E. for 5 miles and then S.S.E. for 3 miles. *Construct* a compass showing N., S., E., W., N.E., S.E. and S.S.E., and also a diagram of the ship's course. How far will she be, direct, from her starting-point, and on what bearing must she sail to return to her starting point ?

* **39.**

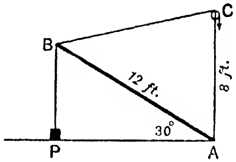
	Length in Yards.	Bearing in Degrees.
16th Hole - - -	520	95
16th Hole to 17th Tee -	40	190
17th Hole - - -	170	235
17th Hole to 18th Tee -	30	200
18th Hole - - -	390	285

The above are the details of the last three holes of a golf course. Draw a correct diagram. [See next page.]

Some spectators at a golf match saw one couple drive off at the 16th tee and then went direct to watch another couple holing out on the 18th green. As a matter of fact, how far is it direct from the 16th tee to the 18th hole ?

Miscellaneous.

* * **40.** AB (12 ft.) represents the jib of a crane, which can turn in a vertical plane about A. It is raised and lowered by means of a rope BC, which passes over a pulley at C (8 feet vertically above A). A chain BP hangs from B and is attached to a weight at P.



The angle the jib makes with the ground is originally 30°. Find in any manner (1) through what angle AB will have to turn, (2) by how much BC will have to be shortened, in order that the weight P may be raised 3 feet from the ground. Draw a careful figure showing P in its final position.

* * **41.** A road ABCD consists of three portions, AB running due E., BC due S. and CD due E. BC is 100 yards long. A church tower (T) lies in CB produced. From a point in CD 100 yards from C the bearing of T is N. 25° W. How far is T from B, and what is the bearing of T from a point in AB 100 yards from B ?

* * **42.** A tunnel is to be made through a hill direct from A to D. Outside the hill the following measurements are recorded :

Lap.	Distance.	Bearing.	Slope.
	Yards.	Degrees.	Degrees.
AB	470	90	0
BC	350	40	+2
CD	95	316	+3

Find (1) the distance between A and D direct, (2) the bearing of the line AD, and (3) the slope of the line AD.

* * 43. Two observers, B and C (of whom B is 3000 yards due north of C), telephone the position of a Zeppelin at the same time. From B its direction is due east and its elevation 12°; from C its bearing is 78°. Find its height and its horizontal distance from B.

* * 44. A, B are respectively north and east of a tower whose height is 100 feet and are in the same horizontal plane as the base of the tower. If the angles of elevation of the tower from A and B are 18° and 24°, find the distance and bearing of B from A.

EXAMPLES 4 b (CALCULATIONS).

Angles.

* 1. What are the complements and supplements of (i) 51° 28′ 38″ and (ii) 12° 34′ 56″ ?

* 2. Two objects are in directions at right angles. The bearing of the left-hand object is 198° 57′ 46″. What is the bearing of the right-hand object ?

* 3. A is a town 5 miles due north-east of B ; C is a town 5 miles from both A and B : find (by drawing to scale) two possible positions for C, and name them C and D.

Measure CD and the angles which CA and DA make with the north-and-south line.

Obtain the size of these angles also by calculation, showing your working clearly.

Mariner's Compass.

* 4. What direction is opposite to S.S.W. ? What directions are at right angles to it ?

* 5. How many degrees is it between the directions N.E. and E.N.E. ?

* 6. How many degrees is it between the directions N.N.E. and S.S.W. ?

* 7. What is the number of degrees between the directions E. by N. and E. by S. ?

* 8. Can you say the names of all the 32 points of a Mariner's Compass in order in a minute ? (Of course without reading them.) [To say the names in order is called "Boxing the Compass."]

Latitude.

*9. If Petrograd is in 60° N. lat., how many degrees is it from the N. Pole? How many degrees is it from the S. Pole? Assuming 1° is approximately 70 miles, give these distances in miles too.

*10. If a place in Tasmania is in 43° S. lat., how many degrees is it from the South Pole, and how many degrees from the North Pole?

*11. Edinburgh is nearly due north of Liverpool, and their respective latitudes are 55° 55′ N. and 53° 25′ N. How many degrees are they apart? Assuming 1° of latitude is 70 miles on the Earth's surface, give the distance in miles too.

*12. Observatories at Copenhagen and Rome are respectively in north latitudes 55° 41′ 13″ and 41° 53′ 34″. By how much do their latitudes differ?

*13. The latitudes of the observatories at Greenwich and the Cape of Good Hope are 51° 28′ 38″ N. and 33° 56′ 03″ S. By how much do their latitudes differ?

Longitude and Time.

*14. 15° of longitude mean one hour of time. If it is noon at Greenwich, what is the time at Sydney (Australia), which is in 151° E. longitude?

*15. If it is 9 a.m. at Greenwich, what is the time in San Francisco, 122½° W. long.?

*16. If it is 3 p.m. at the Cape of Good Hope Observatory (18° 28′ 41″ E.), what is the time at Greenwich?

*17. By how much do the local times at Madrid (3° 41′ 15″ W.) and Rome (12° 28′ 45″ E.) differ?

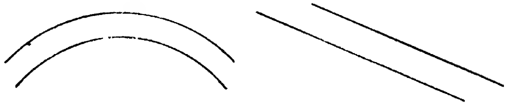
*18. If it is 10 p.m., 31st July, at New York (73° 58′ 30″ W.), what is the time at Athens (23° 43′ 15″ E.)?

*19. It is 2 h. 3 m. 4 s. p.m. at Greenwich, and at a place X it is 11 h. 50 m. 40 s. a.m. What is the longitude of X?

CHAPTER V.

PARALLELS.

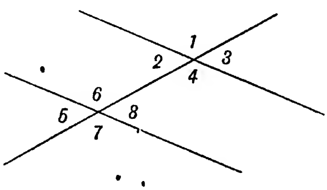
§ **1.** We shall only consider parallel *straight* lines. The word straight is often omitted.



Definition. **Parallel straight lines** are straight lines, in the same plane, which never meet, however far they be produced in either direction.

[The mere fact that two lines can never meet is not enough to make them parallel. They must also be in the same plane, though that plane need not be drawn. The plane can be oblique too.]

If a straight line (it may be called a Transversal) cuts two parallel straight lines it makes 8 angles with them.



Exterior angles, 1, 3, 5 and 7.

Interior angles, 2, 4, 6 and 8.

Corresponding angles, 1 and 6, 3 and 8, 5 and 2, 7 and 4.

Alternate angles, 2 and 8, 4 and 6 (essentially interior angles and essentially on "alternate" sides of the transversal).

(i) *Corresponding angles are always equal.*

(ii) *The sum of the two interior angles on the same side of a transversal is always two right angles.*

(iii) *Alternate angles are always equal.*

The converses (see Chapter VIII.) *are true.*

Briefly (i) Corr.

(ii) 2 int. $\angle$s$=$2 rt. $\angle$s.

(iii) Alt.

The interdependence of these three must be noted, if the truth of any one of them is assumed the other pair necessarily follow.

§ 2. It is possible, **through a given point, to construct a parallel to a given straight line** by copying alternate angles.

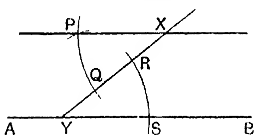
X is the given point.

AB is the given straight line.

Take any point Y on AB.

Join XY.

Copy the angle XYB, at X, and alternately.



[With centre Y, and any suitable radius, get R and S. With centre X, and the *same* radius, get Q. Open your compasses to RS ; transfer the point to Q, and get P.]

[" Good intersections " are necessary for good results.]

N.B.—When you know about parallelograms you should devise a simpler construction.

§ 3. It is also possible, and generally much quicker, neater and more accurate, **to draw parallels, using a set-square.**

[It is much better to slide a set-square along a ruler than *vice versa*.]

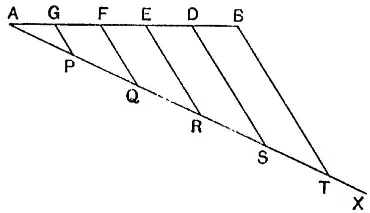
The process is most easily learnt by watching others and with a word-of-mouth explanation. In default of that the process is:

(1) Lay one edge of the set-square along the line to which a parallel is to be drawn. Hold the set-square tight.

(2) Lay a straight edge (ruler) along another edge of the set-square. Hold the straight edge tight.

(3) Loose the set-square and slide it along the straight edge.

§ 4. To divide a given straight line into any number of equal parts. Suppose that it is required to divide AB into 5 equal parts.



Through (either end) A draw a straight line AX.

Cut off 5 equal parts AP, PQ, etc. [The last point T does *not* come on to X. Dividers are convenient for the steps.]

Join TB, and through S, etc., draw parallels to TB, and so get the points D, E, F, G. [Use a set-square for the parallels.]

AB is divided into 5 equal parts.

[If a proof is required use Prop. **10**.]

[It is not very easy to do this division well. The ∠BAX might be about 20° (of course this is judged by eye). Make AX distinctly long enough. Choose the length of the "steps" to get "good intersections." At the end, with dividers "step" along AB to test the accuracy of your work.]

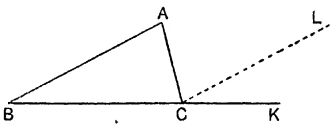
PROPOSITION 1.

§ 5. **General Enunciation.** *If one side of any triangle be produced to prove that the exterior angle so formed must be equal to the sum of the two interior opposite angles, and to prove that the sum of the angles of any triangle must be two right angles.*

Particular Enunciation. ABC is the triangle, whose side BC is produced to K, to prove

(i) $\angle ACK = \angle A + \angle B$;

(ii) $\angle A + \angle B + \angle BCA = 2$ rt. $\angle$s.



Construction. Let CL be parallel to BA.

Proof. (i) $\angle LCA = \angle A$ (alt.),

$\angle KCL = \angle B$ (corr.);

$\therefore$, by addition, the whole $\angle ACK = \angle A + \angle B$.

Q.E.D. (i).

(ii) We have shown $\angle A + \angle B = \angle ACK$;

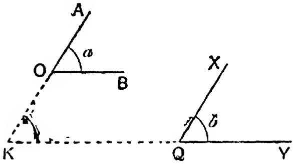
to each add $\angle BCA$;

$\therefore \angle A + \angle B + \angle BCA = \angle ACK + \angle BCA$;

but the latter pair, being adjacent, must make up 2 rt. $\angle$s.

$\therefore \angle A + \angle B + \angle BCA = 2$ rt. $\angle$s. **Q.E.D.** (ii).

§ **6.** It almost goes without saying, if OA and OB are parallel respectively to QX and QY, that $\angle AOB = \angle XQY$.

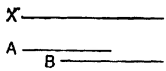


But, for proof, produce AO and YQ to meet at K.

$$\angle a = \angle x \text{ (corr.)}$$
$$\text{and } \angle b = \angle x \text{ (corr.)};$$
$$\therefore \angle a = \angle b.$$

[Appropriate production, etc., is necessary.]

§ 7. Straight lines which are parallel to the same straight line must be parallel to each other.



A and B are both parallel to X, to prove A and B must be parallel.

If A and B were not parallel they would cut, and then there would be two intersecting straight lines, A and B, both parallel to X. And this is contrary to common experience.

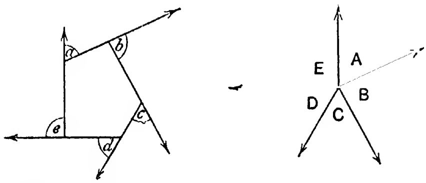
§ **8.** The following proposition is an additional inference to Proposition **1**, and is known as a **Corollary** to Proposition **1.** It was discovered by Robert Simson, about 1700.

PROPOSITION 2.

General Enunciation. *If the sides of a* **convex** *polygon be produced, in the same sense, the sum of the exterior angles so formed must be four right angles.*

[*N.B.*—" In the same sense " or " the same way round " is essential. All ↻ this way round, or all ↺ that way round.]

Particular Enunciation. *abcde* is the convex polygon. To prove that the sum of the exterior angles *a*, *b*, *c*, *d*, *e* is 4 right angles.



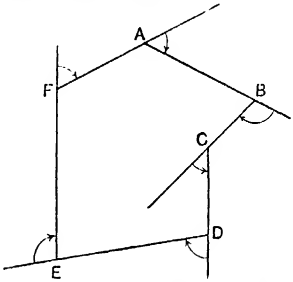
Construction. From some separate point draw lines parallel to the sides of the polygon.

Proof. $\angle a = \angle A$ (corresponding),
$\angle b = \angle B$ („),
etc.;

and so, by addition, the sum of the exterior angles *a*, *b*, *c*, etc., is equal to the sum of A, B, C, etc., and this latter sum is 4 right angles; hence the former sum is equal to four right angles. **Q.E.D.**

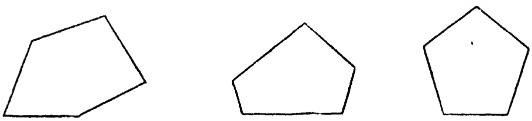
(*N.B.*—A **convex** polygon has no **reflex** angles.)

**** § 9.** *N.B.*—A **convex** polygon implies that there are no re-entrant angles, but provided one is prepared to consider an **angle plus or minus**, according to the way round (right-handed or left-handed), this restriction need not be applied. For instance, in completing the circuit of ABCDEF, and eventually going in the original direction, the turn at C is left-handed, and the turns at the other corners are right-handed. So if the exterior angles at A, B, D, E, F are reckoned *plus* and that at C *minus*, then the total is still 4 right angles.



§ 10. If all the sides and all the angles of a polygon are equal, the polygon is said to be **regular.**

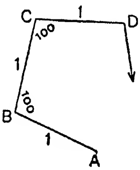
Irregular (sides equal, but angles various).	**Irregular** (angles equal, but sides various).	**Regular** (sides equal and angles equal).

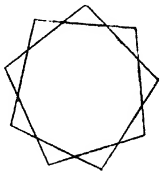


All the exterior angles of any polygon make up 360°. One exterior angle of a regular 5-sided figure (regular pentagon) is 72°. [360÷5=72.] An exterior angle of a regular n-agon is $\frac{360°}{n}$.

[In questions about interior angles of regular polygons work with exterior angles.]

* * § **11.** **When an interior angle of a regular polygon is 100°, how many sides are there?** The exterior angle is clearly 80°, and the solution of the equation $\frac{360}{n}=80$ would give the number of sides $\frac{9}{2}$ or $4\frac{1}{2}$. At first sight this seems impossible. However, if an attempt to draw a regular polygon with sides 1″ and interior angles 100°, by making the side AB, the angle B, the side BC, the angle C, etc., in succession, we should eventually get a join with 9 sides in 2 circuits. Similarly, if n were $\frac{10}{3}$, a regular figure of 10 sides in 3 circuits might be considered.





* § **12.** We have been accustomed to drawing to scale; and, provided the scale chosen requires no calculation (or next to none, in such a simple case as 2 inches to a mile), it is quicker, more accurate, and more convenient to use the wooden scales with which we are provided. However, if the calculations are not merely mental, and if we are concerned with many measurements, it is worth while drawing an accurate scale. The process for the construction of **Plain Scales** is given in the next section (Plain Scales as distinct from Diagonal Scales, considered in Chapter XII.).

There are not different ways for drawing scales under different circumstances. There is only one way. In the following section are given two separate examples, which involve slightly different calculations, but only *one* construction when the calculations are done.

* § **13. Plain Scales.** First calculate the correct length, on the map, of the length you adopt.

(i) If you are told a definite maximum length, an easy sum and no thought, on your part is required.

(ii) If you are not told, you should choose a length that will come out to about half a dozen inches on the map. Simplicity is the main thing [you have to consider that the map may be read in a gale of wind and by indifferent torch-light].

(iii) If you are only given the **Representative Fraction**, or **R.F.** (This fraction simply states how big lengths [not areas] on the map are compared to the same lengths in nature.)

Now consider the following:

The scale 6″ to the mile has R.F. $\frac{6}{1760 \times 36}$, which reduces to $\frac{1}{10,560}$.

[The notation 1 : 10,560 or 1/10,560 is better, for it allows bigger, and clearer, type, or closer lines.]

So when the R.F. is *about* 1 : 10,000, the appropriate amount is 1 mile.

Similarly, when the R.F. is *about* 1 : 20,000, the appropriate amount is 2 miles.

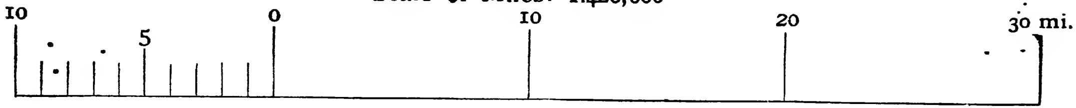
And when the R.F. is *about* 1 : 500,000, the appropriate amount is 50 miles; and so on.

Then, *accurately*, calculate out the length on the map for the appropriate amount.

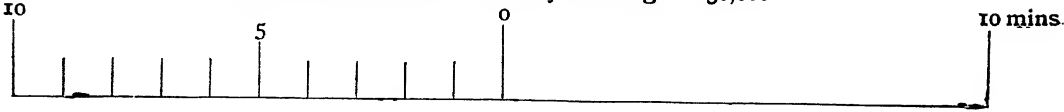
Example (1). For a 1 : 420,000 map the appropriate amount is 40 miles [40, not 42, for the sake of simplicity], and $\frac{1}{420,000}$ of 40 miles works out to about 6·04″.

(iv) Having got the full length of the scale, draw that line, suppose it is 6·04″ [to measure it again, before it is too late, is wise]. Now divide that line up appropriately. Think of it as 40 miles (and forget all about 6·04″). Partly to avoid unnecessary labour, and mainly for simplicity, divide it up into only 4 equal parts, and the left-hand of those parts fully into 10. Number it correctly; see opposite where the 0 comes and that the secondary divisions are numbered backwards. Do not cram in quantities of numbers. The keynote is simplicity. A title is needed.

Scale of Miles. 1:420,000



Scale of Minutes for Cavalry trotting. 1: 30,000



[As a matter of fact, calculating the length of a primary division, etc., and "stepping" along with dividers, is generally more accurately done than an elaborate construction.]

Example (2). As another example of choosing the appropriate amount. "To draw a scale of minutes for cavalry trotting at 240 yards a minute for a 1 : 30,000 map."

Mentally the argument is :

For 1 : 30,000 map the appropriate amount is about 3 miles. 3 miles is somewhat over 5000 yds.

At 240 yards a minute, the time is somewhat over 20 minutes.

Now, *accurately*, 20 mins. trot is $20 \times 240 \times 36$ inches, and on the map this is shown by $\frac{20 \times 240 \times 36}{30,000}$ inches, which works out to 5·76″.

EXAMPLES 5 a (GEOMETRICAL DRAWING).

Parallels.

1. Prick off the points C, E, D (last page). Rule in the straight line ED. Through C construct a parallel to ED. How many inches is it between the parallels ?

2. Prick off the points P, R, S (last page). Rule in the straight line PS. Through R construct a parallel to PS. How many centimetres is it between the parallels ?

3. Prick off the five points P, Q, R, S, T (last page). Rule in the straight line SR. Through the three points P, Q, T make parallels to SR, using the set-square.

4. Prick off the points C, S, Q, P and the extremities of the lines OX and OY (last page). Rule in the lines OX and OY. Using the set-square, through the four points C, S, Q, P make parallels to OX and OY. Do you know the name of the figures formed ?

5. ABC is a triangle in which BC = 4″, CA = 3″ and AB = 2·5″. Produce BC to D, and through C, with the set-square, make a parallel, namely CX, to BA. Measure the five angles A, B, ACX, XCD, ACB in degrees, and in each write its size. Does your figure confirm Proposition 1 ?

Equal division of a straight line.

6. Draw a straight line 5·7″. Divide it into 5 equal parts. Measure one of them in cm. [Use the set-square for drawing the parallels.]

7. Divide a straight line of 4 inches into 7 equal parts. Measure one part in cm.

Bisecting inaccessible angles.

*8. Prick off the points F, E, P, Q (last page). Rule in the straight lines FE and PQ. Construct the bisector of the angle between them. What angle does this bisector make with FE and PQ ?

*9. Prick off the points D, E, P, T (last page). Rule in the two straight lines DE and PT. Construct the bisector of the angle between them. What angle does the bisector make with DE and PT ?

Angles of Polygons.

*10. Prick off the five points P, Q, R, S, T (last page). Rule in the sides of the figure and produce them (of course while you are ruling them and not separately afterwards) in the same sense. (TP produced, PQ produced, etc.) Measure each exterior angle, and in it write its size in degrees. What does your total come to ?

*11. Prick off the five points P, Q, D, F, T (last page). Rule in the sides of the figure. Measure each interior angle, and in it write its size in degrees. What does your total come to ?

**12. Prick off the figure ABCDEF (last page). Measure each exterior angle, and in it write its size in degrees. (Call the exterior angle at C minus.) What does your total come to ?

**13. Treat the figure ABCDEF (last page) similarly to Question 11. (Recollect that the angle at C is reflex and its size is well over 180°.) What does your total come to ?

**14. Draw any large seven-sided figure with at least one reflex angle. Produce the sides in the same sense. In each exterior angle and in each interior angle write its size in degrees, taking especial care about the reflex and minus angles. What do you make their sums ?

Plain Scales.

* **15.** A map is drawn on a scale of 2·5 inches to the mile. State the ratio of any length on the map to the length which it represents (i) as a decimal to two significant figures, and (ii) as a vulgar fraction whose numerator is 1.

* **16.** The scale of a map is 5·4″ long and represents 40 miles. Divide the line up appropriately to show single miles. Figure it neatly, and give its title clearly and R.F.

* **17.** On a map 14 cm. is the length of the scale representing 7 Km. Draw the scale correctly. What is the R.F. ?

* **18.** On a map 320 yards is represented by 3·6″. Draw a scale of yards to read to 500 yards and to show 10 yards. Give the R.F.

* **19.** Draw a scale of feet for a map on which a mile is shown by 5 ft. First calculate by what 500 feet would be shown. Then divide that 500 ft. line up appropriately into 5 equal parts, and the left-hand of those parts into 10 smaller parts. What is the R.F. ?

* **20.** What length does 3 miles occupy on a 1 : 30,000 map ? Draw a scale of miles for it, and divide the left-hand part to show quarters of miles.

* **21.** Draw a scale showing thousands and hundreds of yards for a map whose R.F. is 1 : 30,000. Show 5000 yards. Measure in inches the line which on your scale is 3400 yards.

* **22.** On a piece of unruled paper draw a scale of feet suited to a map on which 6″ represents 1 mile. The greatest distance to be shown is 5000 ft. and the least 100 ft. Give the R.F.

The gradient between two consecutive contours (100 ft. difference) is 1 in 3. Give the distance, on the map, between the contours to the nearest hundredth of an inch.

* * **23.** On a map one mile is represented by 2 inches. Draw a scale of yards for the map. [Draw your scale on a piece of unruled paper.]

Use your scale to draw a line AB to represent 1400 yards, the limit of " effective " rifle fire.

What is the R.F. of the map ?

A road on the map crosses the 200 contour line and the 250 contour line at a distance (on the map) of 0·35″ ; what is the gradient of the road ? The contours are in feet. [Give your result as 1 in . . . .]

* * 24. For a 1 : 40,000 map draw the four following separate scales. In each case show the calculations, which must on no account be so close to the scales that they are not both perfectly easy to read. Neatness is absolutely essential.

Scale of Yards.

Scale of Miles.

Scale of Minutes for Cavalry trotting at 235 yards a minute.

Scale of Seconds for Sound travelling at 1100 ft. a second.

[To check your work from your scales respectively take 4300 yards; $2\frac{1}{4}$ miles; 14 minutes cavalry trot; 12 seconds for sound; and measure them in inches.]

* * 25. Draw a scale of Russian versts suitable for a map whose R.F. is 1 : 40,000 [1 verst = 3500 English feet].

[From your scale measure in inches 2·4 versts, for a check.]

Miscellaneous.

26. Construct a triangle ABC, with BC $=5''$, AB $=4''$, AC $=4\frac{1}{2}''$. Bisect the angle A by AD, cutting BC at D. Through D draw (by construction) DK parallel to BA, cutting AC at K. Measure AK.

27. Draw a triangle ABC, in which BC $=5''$, $\angle$B $=42°$ $\angle$C $=70°$. Divide AB into three equal parts at P and Q, and draw PX, QY parallel to BC, cutting AC at X and Y. Measure PX and QY.

28. Draw a triangle ABC in which BC $=3{\cdot}7''$, A $=43°$, C $=115°$: construct AD, the perpendicular from A on the opposite side, cutting it at D, and bisect the angle C by CE meeting AB at E: let AD, EC (produced if necessary) meet at O, and measure the distance of O from AC.

29. Draw a circle radius 5 cm., centre O. Draw any diameter AOB. Draw a radius OC such that $\angle$BOC $=50°$. Draw CD at right angles to AB, cutting AB in D. Divide OD into 5 equal parts (showing your method). Along OA mark off OE equal to 3 of these parts. Draw EF at right angles to AB, cutting the circle in F. Join OF and measure the angle AOF.

30. Make an equilateral triangle with 2 in. sides. Take any point in one of the sides and draw perpendiculars to the other

two sides. Measure them and show (by measurement) that their sum is equal to the height of the triangle.

31. Two villages, along a straight high road, are 3 miles apart. A railway station is 1 mile from one, and 3 miles from the other. Find (by construction) where I can build a house which will be 2 miles from the station and $1\frac{1}{2}$ miles from the high road. Measure its distance from the nearest village.

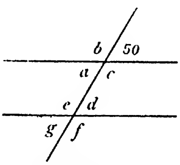
EXAMPLES 5 b (CALCULATIONS).

Some of these questions might be discussed orally. Answers to questions 1-6 *and* 8-12 *are not given.*

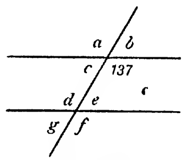
Parallels.

The figures in these examples are not accurately drawn. The numbers give the sizes in degrees of the known angles. You are required to sketch the figures neatly (but by no means necessarily accurately), to calculate the unknown angles, and in each to write its correct size in degrees, and to give in words, briefly, reasons.

1. Give the sizes of the angles a, b, c, etc.



Parallel Lines

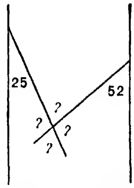


Parallel Lines

2. The oblique line in the letter N makes an angle of 40° with one of the side lines. What angle does it make with the other side line?

3. The letter W is made up of two pairs of parallel lines. On one side the two lines meet at an angle of 37°. At what angle do the two central lines meet?

4. One of the central lines of the letter M makes an angle of 26° with the side line. At what angle do the two central lines meet?



(The two uprights are parallel)

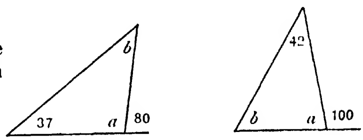
* **5.** Sketch the accompanying figure, and in each of the unknown angles write its size in degrees.

* **6.** Draw two parallel straight lines AB, CD 2 inches apart. Straight lines XY and PQ cut AB and CD at angles of 55° and 26° respectively, and themselves intersect at O, about half way between AB and CD. Through O a line MON is drawn so that ∠MOQ is 38°. Find the angles between all lines that intersect in the figure, and fill in the number of degrees in these angles in the figure. [The lines XY and PQ are to slope different ways.]

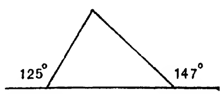
** **7.** At two places A and B, 140 miles apart on the same meridian, simultaneous observations are taken of a star. At A, the northern station, the star appears in the meridian and 29° from the zenith; at B, the southern station, the star is 27° from the zenith. Find approximately the radius of the earth. [The star is so distant that lines drawn from A, from B and from the centre of the earth to it may be considered parallel.]

Triangles.

8. What is the size of $\angle a$ and $\angle b$ in each of these figures ?



9. Calculate the size of the angles of the triangle in the figure.



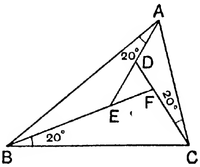
10. Calculate the angles of a triangle if they are in the ratio 5 : 6 : 7.

11. ABC is a triangle, of which ∠A = 40°. ∠ABC and ∠ACB are bisected by lines meeting at I. How many degrees are there in ∠BIC ?

12. In the previous example $\angle IBC$ and $\angle ICB$ are bisected by lines meeting at V. How big is $\angle BVC$?

13. In the accompanying figure prove that the triangle DEF must have the same sized angles as the triangle ABC.

[HINT. In $\triangle DAC$ ext. $\angle EDF = 2$ int. opp. $\angle DAC + \angle DCA$.]



14. Prick off the points T, F, B (last page). Join them up to form a triangle. Measure each angle in degrees, and in it write its size. What do you make the sum total of the three angles ? What ought the sum to be exactly ?

15. Prick off the points P, S, R (last page). Join them up to form a triangle. Produce the side SR. Take a point K on SR produced. Measure the angles PRK, S and P. Is the result as it should be ?

***16.** If two angles of a triangle contain $73° 44'$ and $34° 51$ respectively, what is the measurement of the third angle ?

***17.** If two angles of a triangle contain $76° 54' 32''$ and $100° 23' 45''$ respectively, what is the size of the third angle ?

****18.** ABC is a triangle ; BX bisects the angle ABC, and meets AC at X. If $\angle ABC = 70°$ and $\angle BAC = 50°$, calculate the size of all the other angles in the figure, giving your reason in each case.

****19.** ABC is a triangle with BC produced to D. The angle $ACD = 120°$ and $\angle BAC = 2\angle ABC$. Find the number of degrees in each angle of the triangle.

****20.** The greatest angle of a triangle is three times the least, while the other is 37° less than the greatest : find the three angles, and check your result.

[HINT. Call the least angle $x°$.]

****21.** In a certain triangle ABC, the interior angle A is $\frac{3}{7}$ of the exterior angle at A : and the angle B is $\frac{3}{4}$ of the angle C : find the number of degrees in each angle.

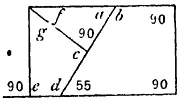
Polygons.

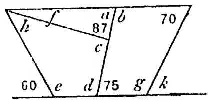
22. ABCD is a quadrilateral.

Find the angles A, B, C, D, given that B is three times A, C is four times A, and D sixty degrees greater than C.

23. The angles of a quadrilateral are in the ratio 3 : 5 : 7 : 9. What are their sizes ?

24. Sketch these figures, and in each unknown angle write its size.





(Two parallel lines)

25. One angle of an octagon is 79°, and the others are all equal to each other. How big are they ?

26. Find the size of each angle of a 9-sided figure, where one angle is a right angle and the rest are all equal.

27. Six of the interior angles of a polygon are each 175°, and the others are all equal and each 169°. How many sides has the polygon got ?

28. A figure has two angles of 164° each and one angle of 173°. If all its other angles are 177° each, how many sides has it ?

29. In the figure ABCDE the angles at A and B are right angles, and the angles at C and E are equal and each is double the angle at D. Calculate the size of each angle in degrees.

* **30.** Prick off the points T, Q, R, S (last page). Join them up to form a four-sided figure. Measure the angles, and in each write its size. What do you make the sum total of these angles ? What ought it to be ?

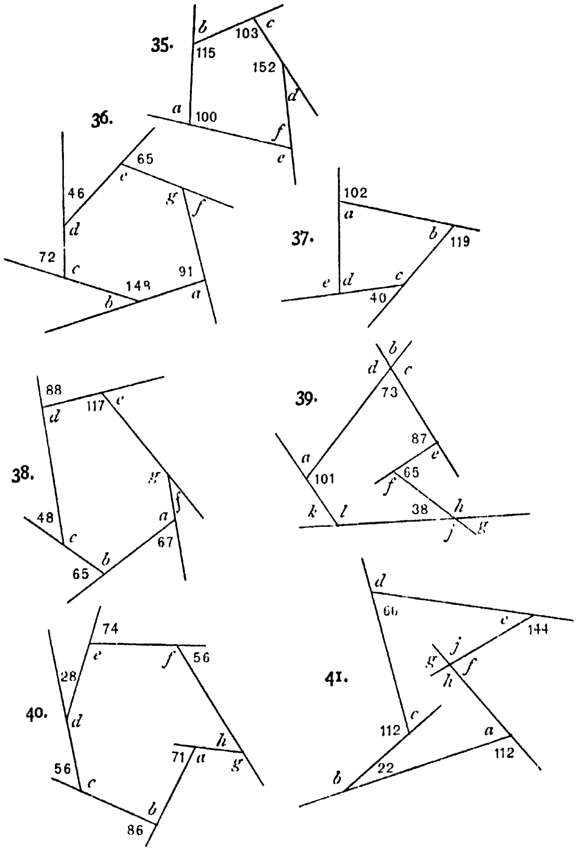
* **31.** What should be the sum of the 5 interior angles of PQRST (last page) ?

* **32.** What ought the sum of the exterior angles of PQRST (last page), if the sides are produced in the same sense, to be ?

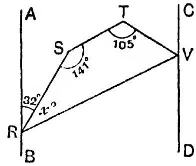
* **33.** What should be the sum of the 6 interior angles of ABCDEF (last page) ?

* **34.** Give reasons for or against the following statement: "There is a certain 5-sided figure with angles 102°, 115°, 93°, 127° and 105°."

*** 35–41.** Sketch the following figures, making them like by eye (and not by measurement). In each angle write its given size or calculated size in degrees.



*42. AB and CD are parallel. If parallels to AB are drawn through S and T, calculate and write down the angles these parallels make with the lines which meet them at S and T respectively, and hence show that the angle TVC $=34°$. Again, if the angle SRV $=x$ degrees prove that the angle RVT $=114-x$ degrees.



Regular Figures.

43. How many sides has a *regular* figure if each of its exterior angles is 24° ?

44. How many sides has a regular polygon if each of its exterior angles is 2° ?

45. The interior angle of a regular polygon is 160°. How many sides must the polygon have ?

46. If the interior angle of a regular polygon be 165°, calculate the number of sides of the polygon.

47. Calculate the size (in degrees) of one interior angle of a regular figure with ten sides.

48. Calculate the size of (i) the interior angle of a regular 15-sided figure, (ii) the number of sides in a regular polygon whose angles are 174°.

**49. Three regular figures with l, m, n sides respectively are cut out in cardboard and placed on a table. It is noticed that when pushed together, they just fill up all the space at their common corner.

Prove $$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=\frac{1}{2}.$$

**50. How many different patterns of regular tiles can be used to make a pavement (if the pavement is made up of tiles of the same pattern) ? Sketch figures showing each.

**51. The size of each exterior angle of a regular n-agon being $\frac{360°}{n}$, draw a graph to illustrate this. [You should take the number of sides on the horizontal scale and the number of degrees in each exterior angle on the vertical scale. How many sides will depend on the size of the squared paper

available, but clearness is the main thing, so do not cramp it. You may join up the points by straight lines.]

* * **52.** The interior angle of the same is $180° - \frac{360°}{n}$. On the same diagram as the preceding (2 graphs on one diagram) draw a graph to illustrate this. [Put titles on the diagram for Questions 51 and 52.]

* **53.** Copy the following table and fill it up completely :

POLYGONS.						
					REGULAR	
No. of sides.	No. of right angles in the sum of the exterior angles.	No. of right angles in the sum of the interior angles.	No. of degrees in the sum of the exterior angles.	No. of degrees in the sum of the interior angles.	No. of degrees in each exterior angle	No. of degrees in each interior angle
3						
4						
5	4	6	360	540	72	108
6						
7						
8						
9						
n						

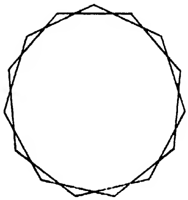
*** * 54.** If one interior angle of a regular polygon is 36°, what size is an exterior angle? How many sides has the figure if you solve the equation $\frac{360}{n}=144$? Draw such a figure. [It is called a pentagram.]

*** * 55.** One interior angle of a regular figure is 72°. Draw the figure.

*** * 56.** If one interior angle of a regular figure is to be 30°, describe the figure, and draw a rough sketch to illustrate your answer.

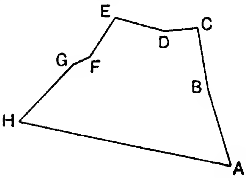
*** * 57.** A man walks a mile, and then turns to the right through an angle of 42°, walks another mile and again turns 42° to the right; and so on, at the completion of every mile, turning 42° to the right. How far does he walk before he returns *exactly* to his starting-point?

*** * 58.** If a regular figure had 15 sides in two circuits, what is the size of its interior angles?



*** * 59.** A surveyor carried out a theodolite traverse from A to H. He observed the separate interior angles to be

A	- -	61°	52′ 59″
B	- -	183	58 16
C	- -	98	02 42
D	- -	198	46 08
E	- -	112	56 23
F	- -	206	44 53
G	- -	162	33 33
H	- -	55	05 54

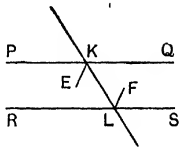


What did he make their sum by observation? What ought their sum accurately to be? If the total discrepancy is distributed equally amongst the eight angles, what correction should be applied to each angle? What is the adjusted value of $\angle A$?

[Only a rough sketch is required; the angles can be obtained correctly from that.]

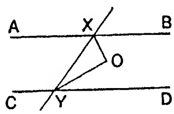
EXAMPLES 5 c (RIDERS).

In a rider the Particular Enunciation, Construction (if there be one) and Proof are required. For the figure a neat sketch is often sufficient. The style of the proof should be like the style of the proof in a proposition.



1. PQ and RS are two parallel straight lines. A straight line KL cuts them. KE bisects ∠PKL. LF bisects ∠KLS. Prove that EK and LF must be parallel.

[HINT. Prove ∠EKL and ∠FLK equal.]

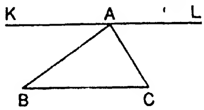


2. A straight line XY cuts two parallel straight lines AB and CD at X and Y respectively; the angles BXY and DYX are bisected by two straight lines meeting at O. Prove that the angle at O is a right angle. [*N.B.*—B and D are on the same side of XY.]

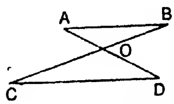
3. AB and CD are two parallel straight lines and GH is another straight line cutting them in G and H. GK is drawn bisecting the angle BGH and HK is drawn at right angles to GK. Prove that HK bisects the angle GHD.

[This is "Converse" (see Chapter VIII.) of preceding.]

4. ABC is any triangle and KAL is a straight line, through A, parallel to BC. Prove that the sum of the 3 angles of the triangle must be two right angles. [The results proved in Proposition **1** are not to be assumed. This is to be another proof of the second part of that proposition.]

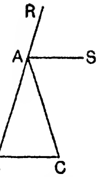


5. ABC is a triangle. BX is perpendicular to CA and CY is perpendicular to BA. Prove ∠ABX = ∠ACY. [HINT. Prove each of these angles is the complement of ∠A.]



6. AB and CD are two parallel straight lines; AD and BC are joined, and meet in O; show that the angles of the triangle AOB are the same as the angles of the triangle DOC.

7. What is the size of one of the angles of an equilateral triangle? Show how to trisect a right angle.



8. ABC is a triangle in which $\angle B$ is given equal to $\angle C$. BA is produced to R. The $\angle RAC$ is bisected by AS. Prove that AS must be parallel to BC.

9. ABCD is a quadrilateral. BO and CO bisect the angles ABC and BCD. If BOC is a right angle, prove that AB is parallel to DC.

10. AB is a given straight line, C a point outside it. Draw CX to cut AB at X, so that it makes an angle of 40° with AB, and give a proof.

11. ABC is a triangle, whose sides BA, CA are produced to E, F, so that AE = AC and AF = AB. Prove that CE and BF are parallel.

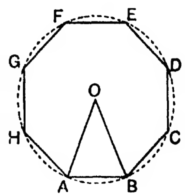
12. ABC is a triangle and AD bisects the angle CAE, BA being produced to E: through C a line CX is drawn parallel to AD, cutting AB at X: prove that AX = AC.

13. ABCD is *any* quadrilateral. CL is perpendicular to AD and DM is perpendicular to BC, neither of which has to be produced. Prove $\angle LCM = \angle MDL$.

14. ABC is a triangle in which $\angle ABC = \angle ACB$. And CK at right angles to BC meets BA produced at K. Prove that

$$\angle ACK = \angle AKC.$$

15. The side BC of a triangle ABC is produced to D. Show that the straight lines which bisect the angles BAC, ACD cannot be parallel.



* **16.** On a given straight line AB to describe a regular octagon. Sketch the figure first and calculate the angles ABC, HAB. In your sketch roughly bisect them to get O. Use O for centre in an accurate figure, etc.

* **17.** PQR is any triangle. The angles Q and R are bisected by straight lines meeting at Z. Prove that $\angle QZR$ is a right angle more than $\frac{1}{2}\angle P$.

* **18.** AB and CD are two straight lines. Construct, without finding their point of intersection, the bisector of the angle between them.

* **19.** ABC is a triangle whose angle A is a right angle. On BC points D, E are taken such that CD is equal to CA, and BE to BA. Show that the angle DAE is half a right angle.

* **20.** The sides BA, BC of a triangle ABC are produced and the exterior angles are bisected by straight lines which meet at D. Prove that the angle ADC is equal to half the sum of the angles BAC and BCA.

* **21.** The side AB of a quadrilateral ABCD is produced to E. The angle BAD contains $\alpha°$, the angle ADC $\beta°$, and the angle CBE $\gamma°$. How many degrees are there in the angle BCD ?

* **22.** ABCD is a quadrilateral in which AB = AD, $\angle$A = $\angle$C = rt. $\angle$. Four pieces of paper are cut out exactly the same size as ABCD ; prove that they can be fitted together to form a square.

* **23.** Straight lines AED, BFE, CDF are drawn within the triangle ABC making the angles DAB, EBC, FCA all equal to one another. Prove that the triangle DEF is equiangular to the triangle ABC.

* **24.** If the bisectors of the angles B and C in the triangle ABC meet in D, by how much does the angle BDC exceed half the angle BAC ?

* **25.** ABC is any triangle in which the angles B and C are bisected by BO and CO. Show that $\angle$BOC is an obtuse angle.

* **26.** ABC is a triangle, and the bisectors of the angles ABC and ACB meet at I. The bisectors of the angles IBC and ICB meet at J. Prove that the angle BJC > 135°.

* **27.** Prove that no convex polygon can have more than three acute angles.

* **28.** Of the exterior angles of a polygon half are equal to those of a regular polygon of m sides, the others to those of a regular polygon of n sides. Find the number of sides of the polygon.

* **29.** AB is a straight line 5 cm. long, and AK a straight line making with it the angle KAB equal to 70°. [Make AK at least 15 cm. long, also produce KA for at least 10 cm.] At the point B in AB, on the same side as AK, the angle ABC is made of varying sizes, 20°, 40°, 60°, 80°, etc., the position of the point C (always on AK) consequently varying.

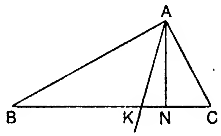
Draw a graph to show how the length of AC varies as the size of the angle ABC varies. [Take 1 cm. for 10° for the horizontal scale.]

What will be the length of AC when the angle ABC is 110°? Give geometrical reasons for your answer.

If the angle ABC is made to increase beyond 110° up to the limit 180°, how do you interpret the lengths of AC? Continue your graph to show the variations in AC, until the angle ABC = 180°.

* * 30. Prove that the angle between the bisectors of two adjoining angles of a quadrilateral is half the sum of the two remaining angles.

* * 31. ABC is any triangle, AN is perpendicular to BC. AK is the bisector of ∠BAC. Prove ∠KAN is equal to half the difference of the angles B and C.



* * 32. The angle which the bisector of one angle of a triangle makes with the exterior bisector of another must be equal to half the third angle.

* * 33. ABC is a triangle in which C > B. AK bisects the angle A and DK bisects the side BC at right angles. Prove that $\angle DKA = \frac{1}{2}(C - B)$.

CHAPTER VI.

CONGRUENT TRIANGLES.

§ **1.** Figures are said to be **congruent** when they are equal in all respects.

Congruence could be settled either (i) (very roughly) by appearance; or better (ii) by direct measurement of corresponding features (but perhaps this might not lead to the detection of small discrepancies); or better still (iii) by direct reasoning, and the various cases are set forth in the following.

If we can settle, by common sense, that a definite number (small for choice) of facts is sufficient to determine a triangle **uniquely**, we have arrived at the number and kind of facts necessary to settle the congruence of two triangles.

In any triangle (besides other things which might be used to determine its size and shape) we have essentially 3 sides and 3 angles. Of these six things it is sufficient to know some to determine the triangle completely in size and shape. A mere attempt to draw a triangle, when but two facts are known, shows us that many triangles are possible fulfilling the two (only) given conditions; but if a third is known too, often (but not always) we can draw the triangle in one way only. This leads us to consider the six following cases by drawing:

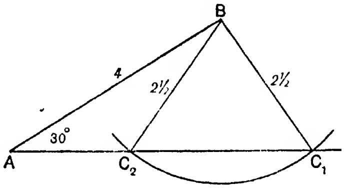
(i) [3 sides.] Draw a triangle with sides 6 cm., 9 cm. and 12 cm. (The possibility should be unique.)

(ii) [2 sides and the included angle.] Draw a triangle with sides 2″ and 3″ and included angle 40°. (The possibility should be unique.)

(iii) [2 angles and the side adjacent.] Draw a triangle with base angles 70° and 60°, and with base 5 cm. (The possibility should be unique.)

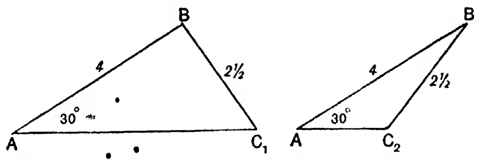
(iv) [2 angles and the side not adjacent.] Draw a triangle with two angles 40° and 60°, and with side opposite the angle 60° of length 3″. First we must recollect that the three angles of any triangle necessarily make up 180°; here we know that two angles make up 100° (*i.e.* 40° + 60°), and so the third angle must be 80°. Secondly, we draw a rough sketch figure on which to mark all the facts known. Thirdly, we can draw the triangle accurately as in case (iii). (The possibility should be unique.)

* (v) [Two sides and a non-included angle.] Draw a triangle ABC in which AB = 4″, BC = $2\frac{1}{2}$″, ∠A = 30°. We may try this as follows. Make ∠A 30° (the arms of the angle had better be some half a dozen inches long). On one arm mark off



AB 4″, and so we have the points A and B all right. Now open the compasses to $2\frac{1}{2}$″, and with centre B determine C on the other arm of the angle. We see on the one figure that there are two possibilities.

The two triangles are also drawn separately.



Both triangles fulfil the given conditions. (This gives rise

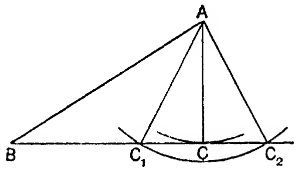
to what is known as the **ambiguous** case, a feature of which is treated in § **6**.)

However, there are exceptions to this. When the angle is a right angle or an obtuse angle the possibility is unique. For instance, draw a triangle PQR in which PQ = 4″, QR = 2″, ∠R = a right angle. Here draw the right angle R first. On one arm cut off QR equal to 2″, then with centre Q and radius 4″ get P (uniquely) on the other arm.

(vi) [3 angles.] This is tantamount to giving only 2 angles, for we know the sum of the 3 angles is 180°, and we have already seen that 2 facts alone are not enough.

Cases (i), (ii), (iii), (iv) are like solving equations of the first degree, where we get only *one* solution.

Case (v) is like solving a quadratic equation where there are *two* solutions. The triangles are ABC_1 or ABC_2. However, if C is a right angle, there is only *one* solution ABC. This corresponds to case of quadratic having *two equal roots.*



§ **2.** It must be noted that the **3 things given must be appropriate.** We might try to draw a triangle with sides 5″, 2″ and 1″, and should fail because the total of the two small sides is not more than the long side. Nor would it do to try to draw a triangle with sides 5 cm. and 7 cm. and included angle 200°, for *all* the angles of any triangle only total 180°. Again, if we attempted to draw a triangle with base 5 cm. and two angles 70° and 120°, we should fail for the same reason.

§ **3.** When we can draw a triangle **uniquely, three appropriate facts are needed,** and we can say that two triangles having those corresponding things equal for any reason (given equal, made

equal, or proved equal) must be congruent. And this leads to the following table:

Case.	Data.	Number of Solutions
1	3 sides	1
2	2 sides and included angle	1
3	2 angles and side adjacent	1
4	2 angles and side non-adjacent	1
5	2 sides and non-included angle	2 (ambiguous case)
	2 sides and non-included right angle	1 (two coincident solutions)
6	3 angles	(insufficient data)

No. 5, the case when the non-included angle is a right angle, is important. We have seen that under these circumstances the triangle is unique. The proposition is considered formally in Chapter IX.

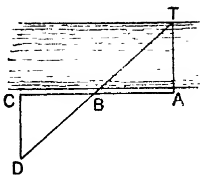
§ 4. In G.D. all these cases may be reckoned equally important. In Riders the case involving 2 sides and the included angle is the most common.

In Surveying, on a large scale (big countries and not mere fields), the case involving 2 angles and a side is almost universal. It is a long process measuring a straight line (perhaps several thousands of yards long) accurately, and perhaps quite prohibitive in the matter of time and cost, if, for instance, forests or dwellings are in the way; but it is *comparatively* easy with good instruments to measure angles, and from that to calculate, trigonometrically, the lengths desired with extraordinary accuracy. Of course, in our drawing we shall do well if we are within $\frac{1}{100}$ inch in a 2-inch line, but mere drawing would not do in accurate work, for $\frac{1}{100}$ inch in 2 inches would mean well over 40 yards in 5 miles, and errors of that sort would be reckoned impossibly large.

In the Ordnance Survey of the United Kingdom bases were measured at Lough Foyle, in the north of Ireland, and on Salisbury

Plain. The whole countries were covered with a network of triangles with those two bases as sides of two of the triangles. All the angles of those triangles were measured and their sides calculated. At a few convenient places the sides were measured too, as a check to the calculation. So that eventually it was possible to calculate the length of one base from the other. The Lough Foyle base was nearly 8 miles long. There turned out to be under 5 inches discrepancy between the measured and calculated lengths of the Salisbury Plain base (which was nearly 7 miles long). To get good results in calculation, just as to get good results in drawing, "good intersections" are essential. In important work in survey no triangle is allowed unless the intersections are good.

§ **5.** As a practical example the **width of a river** may be found thus.



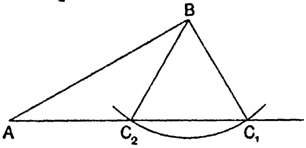
Stand at A opposite some well defined object T (a tree, say). Count the paces to some point, call it B. Put a stick into the ground at B. Count on the *same* number of paces to C. Now turn in a direction parallel to TA. Count the paces till you see D, B, T in the same straight line.

$$CD = TA.$$

[Choose your spots to get "good intersections," and then you can get good results.]

* * § **6.** It is to be noted in the **ambiguous case** that, if two sides about one angle are given, and another (of course, non-included) angle is given too, the **third angles must be equal or supplementary.** An outline of the proof is given here.

For in the accompanying figure the two triangles are ABC_1 and ABC_2.



$\angle AC_2B$ is the supplement of the adjacent $\angle BC_2C_1$.

Now $\angle BC_2C_1 = \angle BC_1C_2$ (isos.);

$\therefore$ etc.

(The angles at the base of isosceles triangle are proved equal in Chapter IX.)

[So that if, for any reason, we know that the third angles cannot be supplementary, we can say the possibilities are unique.]

EXAMPLES 6 a (GEOMETRICAL DRAWING).

3 sides.

1. Draw to scale a plan of a triangular park, whose sides are 420 yds., 370 yds. and 280 yds. Measure the smallest angle.

(*For other examples of triangles with* 3 *given sides, see Examples* 2.)

2 sides and included angle.

2. The hands of a clock are 4 cm. and 2·5 cm. long. Find the distance between their tips when the hands are inclined to each other at 100°.

3. A pair of compasses has legs 3 in. and 3·8 in. long. It is opened so that the angle between the two legs is 41°. What is the radius of the circle which the compasses are now ready to draw ?

4. A stick 3′ 3″ long is upright. It casts a shadow 4′ 3″ long. At what angle is the Sun up ? (*i.e.* what is the altitude of the Sun ?)

5. A wheel has radius 1·7 ft. It has 10 spokes. How far apart are the ends of a consecutive pair of spokes ?

6. A ladder just reaches up to a window ledge 15 ft. from the ground. The foot of the ladder is 6 ft. from the wall (from the window ledge to the ground). Draw a figure to scale, and from it determine the length of the ladder.

2 angles and a side.

7. Draw a triangle ABC, given $BC=3$ inches, $\angle B=35°$, $\angle C=90°$. Measure the remaining sides and angle.

8. A base AB, 300 feet long, is measured. A church spire is at C and the angle CAB is 45°, while the angle CBA is 100°. How far is the church spire from A ?

9. A seashore AB is 3 miles long. A ship is at sea at some point S to be determined. Coast-guards at A and B measure the angles SAB and SBA to be 76° and 54° respectively. The coast-guard stations are in telephonic communication. Plot the triangle SAB to scale. How far is the ship from A ?

2 sides and, non-included, right angle.

10. Without any calculation, construct a right-angled triangle whose longest side is 9 cm. and one of whose other sides is 5 cm. Describe your method of construction, and measure the smallest angle of the triangle.

11. A ladder, 30 ft. long, leans up against a wall. The foot of the ladder is 10 ft. from the wall. How high up the wall does the ladder reach ?

12. The diagonal of a carpet is 22 ft. long. The carpet is 10 ft. wide. What is the length of the carpet ?

Ambiguous Case.

**13. In a triangle ABC, $c=2''$, $b=1''$, $\angle B=20°$. Draw 2 different triangles fulfilling these conditions. Measure a in the 2 cases.

**14. There are 3 villages P, Q and R. $PQ=3$ miles, $PR=2\frac{1}{4}$ miles and $\angle PQR=38°$. How far is Q from R ? (Note the 2 cases.)

3 angles.

15. Draw 2 triangles ABC in which $\angle A=45°$, $\angle B=60°$, and $\angle C=75°$, quite different in size, but naturally of the same shape.

Miscellaneous.

*16. Birmingham is 102 miles from London;

Bristol	,,	107	,,	,,
,,	,,	78	,,	Birmingham;
,,	,,	63	,,	Southampton;
London	,,	71	,,	,

Find from a figure drawn to scale the distance from Birmingham to Southampton.

(Say what scale you are using, and mark all measurements in miles.)

*17. A gun emplacement G is observed from 2 points A and B 100 yds. apart: the angle GAB is measured as 83° and GBA is 76°; find the distance of the gun from A and its shortest distance from AB.

*18. The shadow of a tree is 20 yards long when the sun's altitude is 28°. Find the height of the tree.

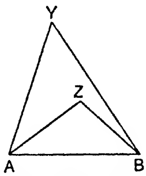
*19. A lawn-tennis court is 78 ft. by 36 ft. Draw it to scale. Measure its diagonal.

*20. The hands of a clock are 2·7 cm. and 1·7 cm. long. How far apart are their tips at 8 o'clock?

*21. A ladder, 30 ft. long, leans up against a house; its base is 12 ft. from the house. Draw an accurate diagram; measure the angle between the ladder and the ground, and the height of the top of the ladder. What will the angle become if the base is moved to double the distance from the house?

*22. The distance YZ is required.

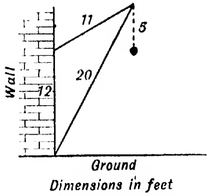
$AB = 540$ ft.,
$\angle YAB = 70°$,
$\angle ZAB = 34°$,
$\angle YBA = 55°$,
$\angle ZBA = 40°$.



Give YZ in feet.

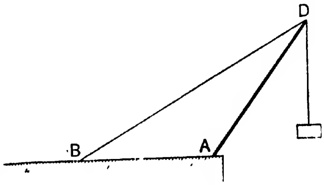
*23. Say which of the following sets of measurements fix the shape and size of a triangle ABC. Draw the triangle ABC when possible, and explain any cases of failure.

(i) $AB = 3''$, $BC = 2''$, $CA = 2\frac{1}{2}''$.
(ii) $AB = 3''$, $BC = 1''$, $CA = 1\frac{1}{2}''$.
(iii) $\angle A = 80°$, $\angle B = 70°$, $AC = 7$ cm.
(iv) $\angle A = 80°$, $\angle B = 100°$, $AC = 7$ cm.
(v) $\angle A = 80°$, $\angle B = 70°$, $\angle C = 30°$.
(vi) $AC = 8$ cm., $BC = 4$ cm., $\angle A = 20°$.



∗ 24. The jib of a crane is 20 ft. long. The tie rod 11 ft. and the vertical is 12 ft. The weight is suspended by 5 ft. of chain. Draw the figure correctly to scale. How high is the weight above the ground ?

∗ 25. In the shear-legs shown in figure, AD is 30 feet, BD is 50 feet and the angle BAD is 130°. The load is supported by a chain passing over a pulley at D and controlled by a winch at A. If the end B of the tie-rod BD is moved away from A until D is brought vertically over A, find (1) the distance through which B is moved, and (2) the length of chain which must be let out so that the load remains at the same height above the ground.

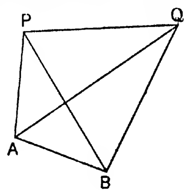


∗ 26. Two vertical posts 3 ft. and 5 ft. high stand on a horizontal plane 4 ft. apart. A peg in a suitable position in the ground is to be connected with their tops by two tight cords each 6 ft. long.

Use a scale of 3 cm. to a foot, and draw carefully the triangle whose vertices are respectively at the peg and the feet of the posts.

Also find, by drawing to scale, the angle between the cords.

∗ 27. The distance between two inaccessible mountain tops, P and Q, is required. Some way off the following measurements were noted :



$$AB = 5{\cdot}3 \text{ miles},$$
$$\angle PAB = 103°,$$
$$\angle QAB = 52°,$$
$$\angle PBA = 38°,$$
$$\angle QBA = 100°.$$

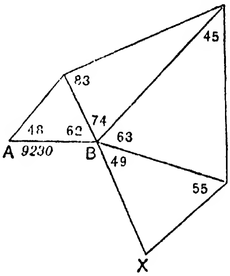
Give PQ in miles.

∗ ∗ 28. A ladder 25 feet long is placed so as to reach a window 24 feet high. If the ladder is turned over to the other side

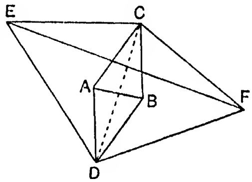
of the street it reaches a point 7 feet high, provided its foot is kept on the same spot as before. Find the breadth of the street.

* * 29. A ladder 30′ long reaches a point 25′ high on a house. On turning the ladder over to the other side of the street it is found that the distance between the top of the ladder in the two positions is 35′. Find, by drawing, how high the ladder reaches, and what is the width of the street.

* * 30. Part of the triangulation of a country is shown on the accompanying diagram, which is not drawn to scale. AB = 9230 feet, and the sizes of the angles observed are given in degrees. How long is BX ?



* * 31. In order to have a base EF of reasonable length for the survey of a big country, the length AB was carefully measured 2345 feet. The base was extended by carefully observed angles. Ignoring the minutes and seconds (which are necessary for trigonometrical calculation, but not for drawing), the angles were



At A	- -	$\angle CAB = 60°$,	$\angle DAB = 85°$.
At B	-	$\angle CBA = 75°$,	$\angle DBA = 50°$.
At C	- -	$\angle ECD = 69°$,	$\angle FCD = 74°$.
At D	- -	$\angle EDC = 68°$,	$\angle FDC = 71°$.

What is the length of EF in feet ? (The figure is only rough.)

[First draw a rough figure like the accompanying, on which to mark the sizes of the angles given. In your correct figure you will find AB and EF cut as a matter of fact.]

* * 32. Two headlands are observed from a ship to bear S.W. and S. 22° 30′ E. respectively; after the ship has sailed S. 22° 30′ W., 7 miles, the bearings become S. 79° W. and S. 67° 30′ E. respectively.

Find the bearing and distance of the one headland from the other. Scale, 1 cm. to 1 mile.

* * 33. From a ship the bearings of two objects A and B are N. 30° E. and N. 20° W. After the ship has sailed 5 miles due N., the bearings are N. 60° E. and N. 40° W. respectively. Find, by drawing to scale, the distance and bearing of A *from* B.

* * 34. From two places, A and B, 500 yards apart, the bearings of a place C are N. 30° E. and N. 48° W. respectively. If the bearing of B from A is N. 70° E., find, by drawing, the distance of C from the line AB.

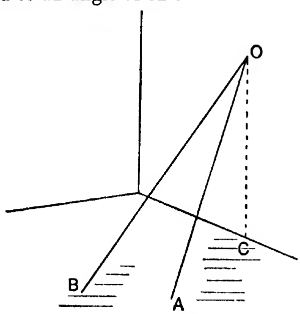
* * 35. A steamer travels at 20 knots through the water; and there is a 5-knot current in the direction N.N.E. What will be the true direction of the steamer, if she steers continuously due east? And in what direction must she steer so as to travel due east?

* * 36. A trap-door 2 ft. 6 in. by 2 ft. in the horizontal floor of a room is opened to an angle of 30°, the hinge being in a long side. What angle does a diagonal make with the horizontal?

* * 37. ABCD is a door, 4 ft. wide and 7 ft. high. The hinges are in the side AB. When shut the edge CD is in contact with the door-post C′D′. Find the distance between C and D′ when the door is opened to an angle of 32°.

* * 38. O is a small hole in the shutter of a dark room. OA represents a sunbeam. After some time the ray has moved to OB. OC is a vertical line meeting at C the floor (shaded horizontally).

AB = 4 feet;
CA = 5 feet = CB;
OC = 6 feet.



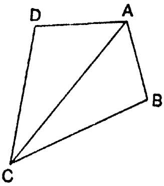
Draw to any convenient scale the triangle AOB. Assume

that the ray revolves round O through 15° in each hour, and find the time taken by the spot of light on the floor to go from A to B.

Also find the angles which OA, OB make with the floor.

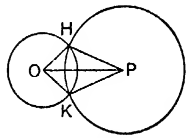
EXAMPLES 6 b (RIDERS).

In a rider the Particular Enunciation, Construction (if there be one) and Proof are required. For the figure a neat sketch is often sufficient. The style of the proof should be like the style of the proof in a proposition.



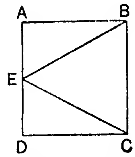
3 sides.

1. ABCD is a quadrilateral in which AB = AD and CB = CD. Prove that the angle ABC must be equal to the angle ADC.



2. Two unequal circles with centres O and P are drawn cutting at H and K. Prove that ∠HPO = ∠KPO.

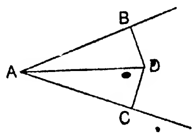
3. C is the middle point of the straight line AB. Through C a straight line CD is drawn. If DA = DB, prove that CD is perpendicular to AB.



2 sides and the included angle.

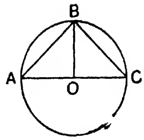
4. ABCD is a square, and E is the middle point of the side AD. Prove BE = CE.

5. If AD be the bisector of the angle BAC, and D be joined to two points B and C on the arms of the angle such that AB = AC, prove DB = DC.

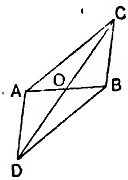


[Note especially that you are *not* entitled to say ∠B = ∠C, both right angles, because they happen to be drawn so. You ought to draw a "general" figure.]

6. Draw a diameter AC of a circle whose centre is O. Draw OB at right angles to AC to meet the circle at B. Prove that the straight lines AB, BC are equal.



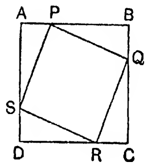
7. Two straight lines AOB and COD bisect each other at O, so that AO = OB and CO = OD. The points A and C are joined, also B and D. Prove that AC = BD. If AD and BC are joined too, prove them equal.



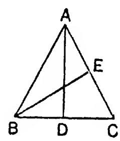
8. ABCD is a square. On its sides are taken the points P, Q, R, S; so that

AP = BQ = CR = DS.

Prove that all the sides of PQRS are equal. [Avoid taking P, etc., anywhere near the mid points of the sides.]

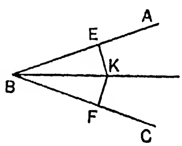


9. ACB is an equilateral triangle; D and E are the middle points of BC and CA. Show that AD = BE.

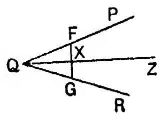


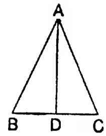
2 angles and a side.

10. ABC is any angle. K is any point on the bisector of the angle. Perpendiculars KE, KF are let fall from K on to BA and BC. Prove that these perpendiculars are equal.

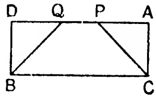


11. PQR is any angle bisected by the straight line QZ. X is any point on that bisector. Through X a straight line FXG is drawn perpendicular to QZ cutting PQ and QR at F and G. Prove FX = XG.





12. If the bisector of the vertical angle of a triangle is at right angles to the base, prove that the triangle must be isosceles (*i.e.* the triangle must have two equal sides).

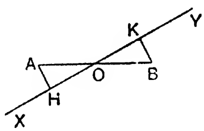


13. DACB is a rectangle. The angles at B and C are bisected by lines which cut DA in Q and P. Prove that BQ = CP.

[*N.B.*—A rectangle has its opposite sides equal and all its angles right angles.]

14. ABC is a triangle in which AB = AC. BL is perpendicular to AC and CM is perpendicular to AB. Prove BL = CM.

[HINT. Consider the triangles BLA, CMA.]

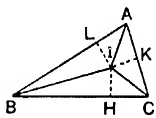


15. Through the middle point O of the straight line AB another line XOY is drawn. Prove that the perpendiculars AH and BK let fall from A and B on XY must be equal.

16. In a triangle ABC the lengths of the perpendiculars let fall from A and B upon the opposite sides are equal; prove that AC = BC.

17. From the ends B, C of one side of an acute-angled triangle ABC, perpendiculars BE, CF are drawn to the opposite sides. If AE is equal to AF, the triangle is isosceles.

rt. ∠ and 2 sides.



18. If the bisectors of the angles B and C of a triangle ABC meet in I, show that AI bisects the angle A.

[HINT. The perpendiculars IH, IK and IL are needed.]

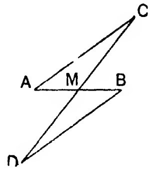
Ambiguous Case.

****19.** ABCD is a quadrilateral such that AB = AD and angle ABC equal to the angle ADC. Prove BC = CD.

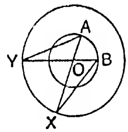
[HINT. ABCD is given a 4-sided figure, so that BCD cannot be a straight line.]

**20. A point D lies inside a triangle ABC in which AB is equal to AC, and the angles DBA, DCA are equal. Prove that DB is equal to DC and that AD bisects the angle BAC.

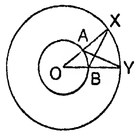
Miscellaneous.



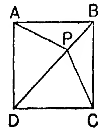
21. Two straight lines AB, CD bisect each other at M. Prove that AC = BD. Also prove that AC and BD must be parallel.



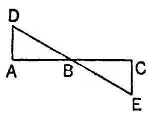
22. Two circles have the same centre O; AO and BO are any two radii of the smaller. AO and BO produced (beyond O) cut the larger circle at X and Y respectively. Join AY and BX. Prove AY = BX.



23. Two circles have the same centre O. Two lines OAX and OBY through O cut the inner circle in A and B and the outer circle in X and Y. Prove AY = BX.



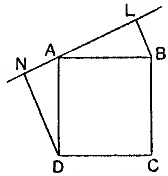
24. ABCD is a square. P is *any* point on the diagonal BD. (Avoid taking P anywhere near the mid-point of BD.) Prove that the triangles ABP and CBP must be congruent.



25. AC is a straight line bisected at B. From A and C, AD and CE are drawn on opposite sides of AC perpendicular to AC and equal to each other.

If B be joined to D and E, prove that DBE is a straight line.

26. Two circles, with centres A and B, cut at X and Y. Prove that AB is perpendicular to XY and bisects it.



27. ABCD is a square and BL, DN are drawn perpendicular to any line passing through A. Prove that AL = DN.

28. ABC is a triangle, E the middle point of AC. Join BE and produce it to F, making EF = BE. Join CF. Prove that the triangles ABC and BFC are equal in area.

29. P is a point between two intersecting lines OA and OB. OP is joined and produced to Q so that PQ = OP: a line through Q parallel to OB meets OA at S, and SP produced meets OB in R. Prove that P bisects SR.

30. P and Q are any two points on the diagonal AC of square ABCD. Prove that the angle PBQ = the angle PDQ.

* **31.** ABC is a triangle; M and N are the middle points of AB, AC; two lines through M and N, perpendicular to AB, AC, meet at O. Prove OB = OC.

* **32.** ABC is an isosceles triangle: from the equal sides AB, AC two equal parts AX, AY respectively are cut off. Prove BY = CX.

* **33.** The sides AB, AC of a triangle are produced to D and E respectively and the bisectors of the angles DBC and ECB meet at X; prove that the perpendiculars from X on AD, AE and BC are all equal.

* **34.** If ABCD be a square and E and F the middle points of AB and BC respectively, prove that AF = DE.

* **35.** ACXY and ABPQ are two unequal squares on opposite sides of the same straight line ABC. Prove QC = BY.

* **36.** In the side AB of the triangle ABC, or in AB produced, a point E is taken so that AE = AC, and in the side AC, or in AC produced, a point D is taken so that AD = AB. Show that BC = ED.

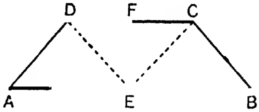
* **37.** ABCD is a rhombus; on its diagonal AC equal distances AM and CN are cut off. Prove that BMDN is also a rhombus.

[*N.B.*—A rhombus is a four-sided figure with all its sides equal.]

* * **38.** With centres A and B circles are drawn intersecting in C and D. If AB meets CD in E, show that the triangles AEC and AED are equal in all respects.

* * **39.** Draw any quadrilateral figure ABCD. Join AC and BD. Prove that if AC bisects the angles BAD and BCD at its extremities it must bisect BD at right angles.

* * **40.** ABCD is a quadrilateral: E and F the middle points of AB and CD. Also EF is perpendicular to both AB and CD. Prove that BC = AD.



* * **41.** ABC is any triangle. Squares ABPQ, ACXY are described, externally, on two of its sides. Prove QC = BY.

* * **42.** ABC is any triangle. Equilateral triangles ABP, ACX are described, externally, on two of its sides. Prove PC = BX.

* * **43.** ABC is a triangle and P the middle point of its base BC; through P a straight line RPS is drawn cutting AB in R and AC *produced* in S. Show that PS > PR.

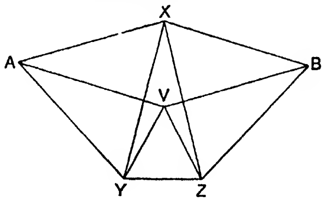
[HINT. Through C draw a parallel to AB.]

* * **44.** M, N are the middle points of the sides AC, AB respectively of the triangle ABC: BM is produced to D and CN to E, so that MD = BM and NE = CN; prove that DE passes through A.

* * **45.** If ABC is a triangle, squares described externally on AC, AB are ACFH, ABDE; and HB, EC are joined, show that HB, EC intersect at right angles.

* * **46.** YXZ is an isosceles triangle, in which XY = XZ. YAX, YVZ and ZBX are equilateral triangles. Prove that XAVB is a rhombus (that is, that all its sides are equal).

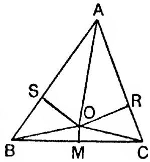
[HINT. Prove ΔVAY congruent to ΔZXY.]



* * **47.** CD is a straight line and G any point in it. On CD and CG on opposite sides the squares CDAB, CGFE are described. A point X is taken in BC such that BX = CG and AX and XF are joined. Show that the figures ABX, XFE and AXFGD may be fitted together to form a square.

****48.** Detect the **fallacy** in the following statement and reasoning:

Any triangle must be equilateral.



Particular Enunciation. ABC is any Δ, to prove it must be equilateral.

Construction. AO bisects the angle A.

OM bisects the side BC at right angles.

Let O be the point where AO and OM meet.

OB and OC joined.

OR and OS perpendicular to AC and AB.

Outline of proof.

(i) Congruent Δs ASO, ARO; and, in particular, AS = AR and OS = OR.

(ii) Congruent Δs OMB, OMC; and, in particular, BO = CO.

(iii) Congruent Δs OBS, OCR; and, in particular, BS = CR.

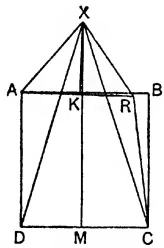
From (i) and (iii), by addition, AB = AC.

Similarly AC = BC;

$\therefore$ AB = BC = CA or every Δ must be equilateral.

****49.** Detect the **fallacy** in the following statement and reasoning:

A right angle must be an acute angle.



Particular Enunciation, etc.

ABCD is a square. CR inside the square = side of square.

Perpendicular bisector of CD is MX.

Perpendicular bisector of AR is KX.

XA, XR, XD, XC joined.

Outline of proof.

(i) Congruent Δs XAK, XRK; and, in particular, XA = XR.

(ii) Congruent Δs XDM, XCM; and, in particular, XD = XC.

(iii) AD = RC (given).

(iv) Using (i), (ii), (iii) we see Δs XAD, XRC must be congruent; and, in particular, $\angle$XDA = $\angle$XCR.

(v) Congruent Δs XDM, XCM; and, in particular, $\angle$XDM = $\angle$XCM.

Adding results (iv) and (v), $\angle$ADC = $\angle$RCD.

Hence right angle = acute angle.

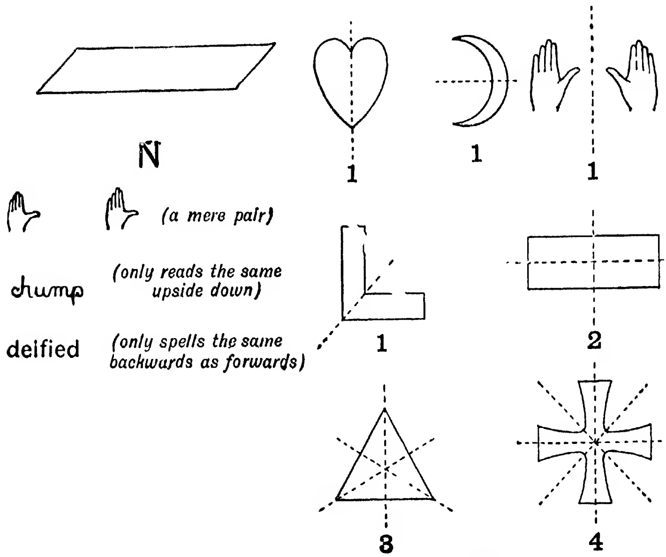
CHAPTER VII.

SYMMETRY.

§ 1. Symmetry about a line (called an axis of symmetry).

Unsymmetrical about a line in the plane of the paper.

Symmetrical about a line, or lines, dotted in the figures; and the number of axes of symmetry.



&c.

If we write something in ink, and blot it with a clean sheet of blotting paper, the original and the impression can be put side by side to form a symmetrical figure.

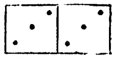
If we write something and hold a **looking glass** by its side, the original and the image in the looking glass might form a symmetrical figure.

Folding a piece of paper and cutting out any shape in the doubled paper (preferably keeping the crease intact), and then unfolding, will show a symmetrical figure.

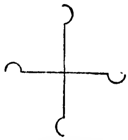
The essential thing is that the two parts should exactly coincide when folded across the axis of symmetry.

§ 2. **Symmetry about a point (central symmetry).**

Examples :



N S Z

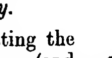


The point should be considered as an axis perpendicular to the plane of the paper.

One part can be made to coincide with another by twisting, or rotating, about a point.

§ 3. A figure may have **symmetry of both kinds.**

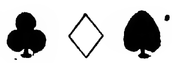
Examples : H O X



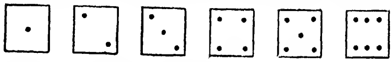
EXAMPLES 7 (GENERAL).

In the following a definite numerical answer is seldom asked, but the "Answers" will afford some check to your work. Some questions might be discussed orally.

1. Sketch these figures, indicating the axes of symmetry by dotted lines (and writing below a number to give the total number of axes of symmetry) :



2. These figures represent the faces of one of a pair of dice. Sketch each and treat it as in Question 1:



3. Discuss the symmetry of the four signs

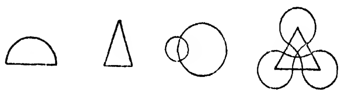
+ − × ÷

4. In how many ways are the signs = and ∴ symmetrical? Give sketches to illustrate your answers.

5. Discuss the symmetry of the signs

‖ † * §

6. Sketch these figures, and indicate the axes of symmetry by dotted lines:



7. Write out the whole alphabet in the simplest possible block capitals (A --B--, etc.) and with dotted lines indicate the axis

1 1

(or maybe axes) of symmetry where they exist, putting numbers or nil below to say in how many ways each is symmetrical.

[Symmetry about an axis in the Plane of the Paper only is to be considered.]

8. Write down the Roman numbers of a clock face (I, II, III ... to XII), and with dotted lines show the axes of symmetry.

9. Write down the Arabic numerals (1, 2, ... to 9, 0). Which are symmetrical? Indicate the axis of symmetry by a dotted line.

10. The word SWIMS reads the same upside down. Discuss the symmetry.

11. Discuss the symmetry of each of the letters SOS, and of the combination of letters, and of their Morse Code equivalent

• • • — — — • • •

***12.** Fold a piece of paper across once. With a pair of scissors cut out any shape on the doubled paper, but leaving the fold intact. Unfold it, and with a pencil show the outline of a figure symmetrical with regard to one axis. (In your cutting take care to avoid a shape which is more properly called symmetrical with regard to more than one axis.)

*13. Fold a piece of paper once, and again at right angles. With a pair of scissors cut out any shape on the quadrupled paper, but leaving the fold intact. Unfold it, and with a pencil show the outline of a figure symmetrical with regard to two axes.

*14. Invent five separate figures symmetrical with regard to 1, 2, 3, 4 and 5 axes respectively.

*15. ABCD is a parallelogram (AB 9 cm., BC 4 cm., and with angle B 50°). P, Q, R and S are the middle points of AB, BC, CD and DA respectively. Draw correctly the appearance of the figure if bent about SQ and also, separately, about PR.

*16. Fold a piece of paper once, and then fold it on a 60° line and 120° line, so that all the creases meet at one point and there are six thicknesses of paper. Cut out any shape, and then unfold the paper. In how many ways is the figure symmetrical? With a pencil trace its boundary and indicate the axes of symmetry by dotted lines.

**17. Make out a table like the following, and indicate whether there is point-symmetry or line-symmetry in each one of a pack of cards. In the case of line-symmetry, indicate the number of axes.

	♠	♡	♢	♣
Ace				
King				
Queen				
Knave				
10				
9				
8				
7				
6				
5				
4				
3				
2				

CHAPTER VIII.

CONVERSES.

§ **1.** When two propositions are related so that what is to be proved in one is given in the other, and *vice versa*; then one proposition is said to be the **converse** of the other.

Because a proposition is true, it is no argument to say its converse must be true also.

Consider the following pairs:

1. (*a*) *In any triangle, if one side be given equal to another, then the angles opposite to them must be equal.* (**True.**)
   (*b*) *In any triangle, if one angle be given equal to another, then the sides opposite to them must be equal.* (**True.**)

2. (*a*) *The train passed two consecutive mile-posts in a minute, and, hence, must have travelled on an average* 60 *miles an hour for some time.* (**True.**)
   (*b*) *The train was travelling at* 60 *miles an hour for some time, and, hence, would have passed two consecutive mile-posts in a minute.* (**True.**)

3. (*a*) *If the sides of one triangle are equal to the corresponding sides of another triangle, then the corresponding angles must be equal.* (**True.**)
   (*b*) *If the angles of one triangle are equal to the corresponding angles of another triangle, then the corresponding sides must be equal.* (**Untrue.**)

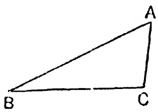
4. (*a*) *It is raining, therefore the roof of the house is wet.* (**True.**)
   (*b*) *The roof of the house is wet, therefore it is raining.* (**Untrue.**)

§ 2. Many (but by no means all) converse propositions in Geometry are proved by the **Reductio ad absurdum** method. One shows, by considering a contrary *conclusion* (the *data* are never questioned) that an ABSURDITY is arrived at, and hence one argues that the contrary conclusion is wrong.

If there is more than one contrary possibility, one exhausts those possibilities in turn, and the name **Proof by Exhaustion** is given.

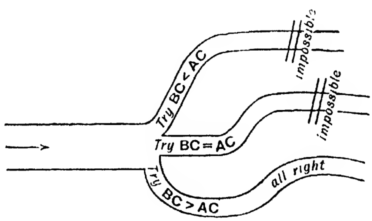
To prove A = B.	To prove X is greater than Y.
2 Possibilities.	**3 Possibilities.**
Either A is equal to B,	Either X is greater than Y,
or A is not equal to B.	or X is equal to Y,
	or X is less than Y.

An illustration of the *Reductio ad absurdum* method of proof (used in Prop. **7**) is as follows:



Given $\angle A$ is greater than $\angle B$.
To prove BC is greater than AC.

There are 3 possibilities. Take them in turn.



(i) Suppose BC is less than AC.

[" Suppose " is equivalent to choosing a road to try.]

∴ $\angle A$ is less than $\angle B$ (by some previous proposition), which is impossible, for $\angle A$ was given greater than $\angle B$.

[Whenever you prove anything impossible, look back to where you last said "Suppose." The last thing you supposed must be wrong; that is, the road you tried is blocked by an impossibility.]

$\therefore$ BC is not less than AC.

(ii) Suppose $BC = AC$.

[We are trying the middle road.]

$\therefore$ $\angle A = \angle B$ (by some previous proposition), which is impossible, for $\angle A$ was given greater than $\angle B$.

$\therefore$ BC is not equal to AC.

Hence we conclude from the results of (i) and (ii) that

BC is greater than AC.

[There are only three possibilities, and of these, two have proved failures, so that the third must be all right.]

§ 3. **Many important discoveries are made by wondering whether converses of quite obvious propositions are true.** If the Earth spins and the stars stay still, the stars must appear to cross the sky. This made Copernicus wonder, "As the stars appear to cross the sky, does the Earth spin?" Copernicus thought "Yes," and, since then, experiments with tops and pendulums have proved that he was right.

Glaciers carry stones with them, which scrape parallel grooves on the rocks over which they pass. This suggested to geologists, "Is the converse true, that rocks on which parallel grooves are found have been scraped by glaciers?" Further investigation has led them to the view that this is so.

It has often been noticed that people bitten by certain mosquitoes get malaria. What is the converse of this proposition, and what use has been made of it?

EXAMPLES 8 (GENERAL).

[*N.B.—These examples might be discussea orally. Answers are not given.*]

*1. What is the enunciation of the converse of the following proposition?

"If the men on this island are niggers they must be black men." Is the converse true ?

* 2. Write out the enunciation of the converse of :

"It is raining, and therefore there are clouds overhead." Is the converse true ?

* 3. What is the converse of "There is a hole in the lawn-tennis net, and therefore the ball can go through" ?

* 4. "A motor-car passed two consecutive milestones in 2 minutes, and therefore must have been travelling, on the average, at 30 miles an hour." What is the converse ?

* 5. In Algebra we argue thus :

$$\text{If} \quad 2x = 10,$$
$$\text{then} \quad x = 5.$$

What is the converse argument ; is it sound ?

* 6. What are the enunciations of the converses of the following propositions ? Granted the truth of the original, is the converse necessarily true ?

(a) The sum of the three angles of a plane triangle must be two right angles.

(b) If the opposite sides of a quadrilateral are given parallel, its diagonals must bisect each other.

* 7. State the converse of :

"I have pulled the chain (in a railway carriage) to communicate with the servants of the company, and therefore I am liable to a fine of £5."

* 8. State the enunciations of the converses of the two following propositions :

(a) When the sun is at its highest, it must be mid-day.

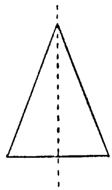
(b) This book is on geometry, and therefore I hate it.

Are the converse propositions true ?

CHAPTER IX.

ISOSCELES TRIANGLE.

§ **1.** A triangle with two sides equal is called an **isosceles triangle.**



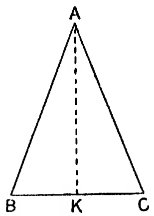
N.B.—An isosceles triangle is symmetrical as indicated; and, if the truth of this statement is assumed, the two following propositions would be omitted (but by no means the examples thereon). There is an advantage in knowing these propositions thoroughly, in that they show the style in which proofs are required.

PROPOSITION 3.

§ **2. General Enunciation.** *The angles at the base of an isosceles triangle must be equal.*

Particular Enunciation. ABC is the isosceles triangle, in which AB = AC. To prove ∠B = ∠C.

Construction. Let AK be the bisector of ∠A, and let AK cut BC at K.



Proof. In the △s ABK, ACK,

$$\because \left\{ \begin{array}{l} \text{AB} = \text{AC (given),} \\ \text{AK (common),} \\ \angle\text{BAK} = \angle\text{CAK (const.);} \end{array} \right.$$

∴ △s ABK and ACK are congruent;

and, in particular, ∠B = ∠C. **Q.E.D.**

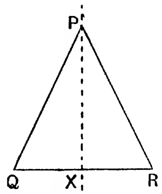
§ **3.** The following proposition is the **converse** of the preceding, and it is as well to read the chapter on converses (Chapter VIII.) in this connexion.

PROPOSITION 4.

General Enunciation. *If two angles of a triangle be equal, then the sides which are subtended by them must also be equal.*

Particular Enunciation. PQR is the triangle, in which $\angle Q = \angle R$. To prove $PQ = PR$.

Construction. Let PX be the bisector of $\angle P$, and let PX cut QR at X.



Proof. In the $\triangle$s PQX, PRX,

$$\begin{cases} PX \text{ (common)}, \\ \angle Q = \angle R \text{ (given)}, \\ \angle QPX = \angle RPX \text{ (const.)}; \end{cases}$$

$\therefore$ $\triangle$s PQX and PRX are congruent;

and, in particular, $PQ = PR$. **Q.E.D.**

§ **4.** It is important that **the bisector of the vertical angle of an isosceles triangle also bisects the base at right angles.** Many calculations, with regard to isosceles triangles, use this fact.

§ **5.** The most important isosceles triangle is an **equilateral triangle,** which has all three of its sides equal.

§ **6.** Frequently, when two triangles have 2 sides of the one equal to 2 sides of the other, each to each, and a pair (*but non-included*) of corresponding angles equal, the triangles need not be congruent; but when these non-included angles happen to be right angles the triangles are necessarily congruent. This is a most important fact (referred to in Chapter VI. page 79), and a formal proof is given here.

PROPOSITION 5.

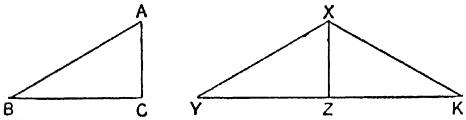
General Enunciation. *If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.*

Particular Enunciation. XYZ and ABC are the right-angled triangles, in which $\angle XZY = \angle ACB$, both right angles,

hypotenuse $XY = AB$,

side $XZ = AC$,

to prove $\triangle XYZ \equiv \triangle ABC$.



Construction. Take up the $\triangle ABC$ and apply it to the $\triangle XYZ$, so that the point A is on the point X, and the straight line AC along XZ, so that the point C falls on the point Z (for $AC = XZ$), and turn the $\triangle ABC$ over so that it takes up the position XKZ.

Proof. (i. YZK a st. line.)

$\because$ $\angle XZY$ and $\angle XZK$ *are right angles and are adjacent*,

$\therefore$ YZK is a straight line;

$\therefore$ XYK *is a triangle, and it is isosceles.*

(ii. $\triangle XYK$ isosceles.) $\because$ $XY = XK$ (XK is the same as AB),

$\therefore$ $\angle XYK = \angle XKY$ (or $\angle XYZ = \angle XKZ$).

(iii. Congruent $\triangle$s.) In the $\triangle$s XYZ, XKZ,

$$\because \begin{cases} \angle XYZ = \angle XKZ \text{ (proved)}, \\ \angle XZY = \angle XZK \text{ (rt. } \angle\text{s)}, \\ XZ \text{ is common}; \end{cases}$$

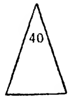
$\therefore$ $\triangle XYZ \equiv \triangle XKZ$.

But $\triangle XKZ$ is simply $\triangle ABC$ in another position;

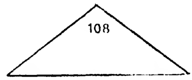
$\therefore$ $\triangle XYZ \equiv \triangle ABC$. **Q.E.D.**

EXAMPLES 9 a (CALCULATIONS).

[*N.B.—Some of these questions might be discussed orally. To such the letter* O *is prefixed.*]



O **1.** The vertical angle of an isosceles triangle is 40°. What must be the size of each of its base angles ?



O **2.** The vertical angle of an isosceles triangle is 108°. What must be the size of each of its base angles ?

O **3.** One angle of an isosceles triangle is 132°. What is the size of each of the other angles ?

O **4.** One angle of an isosceles triangle is 56°. What is the size of each of the other angles ? [Two cases.]

O **5.** In the letter **A** the vertical angle is 42°. What is the size of each of the other four angles ?

O **6.** What is the size of each interior angle, and of each exterior angle, of an equilateral triangle ?

O **7.** In a triangle two of the sides are 8·8 cm. long, and the angle between them is 78°. How big are the other angles ?

8. Draw a triangle ABC, given BC = 4 inches, $\angle B = 74°$, $\angle C = 32°$. Calculate $\angle A$. What can you say about the sides of the triangle, and why ?

9. A ferry boat is moored by a rope 90′ long to a point in midstream. What angle does the rope turn through as the boat crosses from side to side ? The width of the river is 30 yards.

10. If one angle of an isosceles triangle is 126°, what is each of the others ?

11. The base of an isosceles triangle is produced and an exterior angle is 100°. Sketch the figure, and in each angle of the triangle write its size.

12. The base BC of an isosceles triangle ABC is produced to D. If the angle ACD = 115°, find each angle of the triangle in degrees. Give your reasons carefully.

13. One of the equal sides of an isosceles triangle is produced and the exterior angle is 150°. Sketch the figure, and in each angle of the triangle write its size.

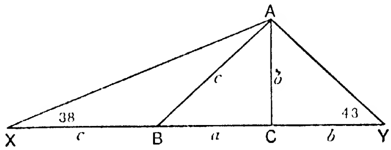
14. ABC is a triangle having $AB = AC$. BC is produced to X and BA to Y. The angle $ACX = 104^\circ$. Calculate the number of degrees in the angle CAY.

15. What are the sizes of the angles of an isosceles triangle in which the base angles are double the vertical angle ?

16. What are the sizes of the angles of an isosceles triangle in which the base angles are each half the vertical angle ?

17. An isosceles triangle has one angle three times each of the others. Calculate the size of each angle.

18. The base BC of a triangle ABC is produced both ways ($XB = BA$ and $CY = CA$). What are the sizes, in degrees, of the three angles of the triangle ABC ?



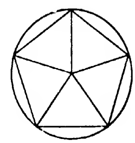
19. ABC is a triangle having $\angle A = 58^\circ$, $\angle B = 76^\circ$, $\angle C = 46^\circ$. In the side BC a point D is taken such that $BD = BA$. Find the number of degrees in all the angles of the triangles ABD, ACD.

Also BE is drawn perpendicular to BC to meet DA produced in E. What are the angles of the triangle ABE ?

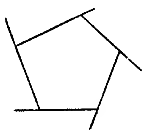
20. D is a point in the side AB of a triangle ABC so that $DB = DC = 3$ in. and the angles ABC and DCA are 53° and 37° respectively ; construct the triangle and measure the sides.

* **21.** ABCD is a quadrilateral in which $AB = AD$, and the diagonals AC, BD meet in O. If angle $BOC = 74^\circ$ and angle $ABD = 26^\circ$, calculate the size of the angle DAC, and show how your result is obtained.

* **22.** (*a*) Imagine a regular pentagon (5 sides) in a circle. What is the size of each central angle ? and also of the base angles of each isosceles triangle ? and hence what is the size of one interior angle of a regular pentagon ?

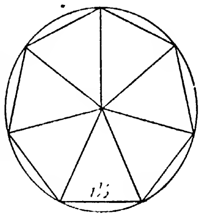


(*b*) Again, imagine each side of a regular pentagon produced. What is the total of all the exterior angles ? What is the size of one exterior angle ? and hence what is the size of one interior angle of a regular pentagon ? [Do the results of 22 a and 22 b confirm each other ?]



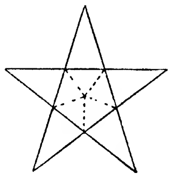
* 23. What is the size of each interior angle of a regular octagon (8 sides)? [Work it out in two ways.]

** 24. On a given straight line, $1\frac{1}{2}''$ long, describe a regular heptagon (7 sides). [A possible plan will be to calculate the base angles of one of the isosceles triangles to give the centre of the circle, and then to "step" round with a pair of dividers. It will need considerable accuracy to draw the regular heptagon well. In theory it is possible to describe other regular polygons in a similar way, but it is not always the best way, and results are apt to be disappointing.] What are the sizes of the base angles of the isosceles triangles?



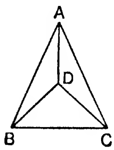
** 25. On a given straight line, $1\frac{1}{2}''$ long, describe a regular hendecagon (11 sides). What are the base angles?

** 26. Describe a "pentagram." [First calculate the various angles indicated by the dotted lines; of what size are the angles?]

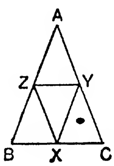


EXAMPLES 9 b (RIDERS).

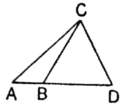
1. Two isosceles triangles ABC, DBC are on the same base BC. Show that the triangles ADB, ADC are equal in all respects.



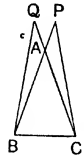
2. Prove that the triangle formed by joining the middle points of the 3 sides of an isosceles triangle is isosceles.



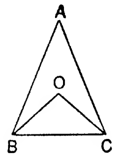
3. The side AB of a triangle ABC is produced to D. BD is made equal to BC. Prove that the angle ABC is double of the angle ADC.



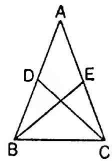
4. The equal sides BA, CA of an isosceles triangle are produced beyond the vertex to P and Q so that AP = AQ. Show that BQ = CP.



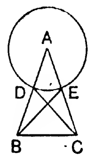
5. ABC is an isosceles triangle in which AB = AC: the angles B and C are bisected by 2 lines which cut at O: prove BO = CO.



6. ABC is an isosceles triangle, and D and E are the respective middle points of the equal sides AB, AC. Prove that CD = BE.

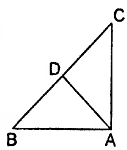


7. Draw an isosceles triangle ABC, in which AB = AC, and a circle with centre A cutting AB at D and AC at E. Prove that CD = BE.



8. D is the middle point of the side BC of a triangle ABC. The triangle is such that the angle DBA = angle DAB: prove that the angles DAC and DCA are equal also.

[Recollect that there is no reason to suppose that the angles at D are right angles They need not be, so it is wiser to draw a more general figure to avoid a danger.]



9. Show that the straight line which joins the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base, and also bisects the vertical angle.

10. ABC is an isosceles triangle having AB = AC. If X and Y are points on BC such that BX = CY, prove that AXY is an isosceles triangle.

11. ABC is an isosceles triangle having AB equal to AC. K is the middle point of the base BC. From BA and CA respectively equal lengths BE and CF are cut off. Join EK and FK. Prove that EK = FK.

12. How many degrees are there in an angle of an equilateral triangle? The angles ABC, ACB of an equilateral triangle are bisected by straight lines which meet in O. How many degrees are there in the angle BOC?

13. ABC, ABD, ADE are equilateral triangles; prove that E, A, C are in a straight line.

14. D, E, F are the middle points of the sides BC, CA, AB respectively of a triangle ABC. Prove that if AB = AC, then DF = DE.

15. ABC is an equilateral triangle. P, Q, R are points in the sides AB, BC, CA, such that AP = BQ = CR. Prove that the triangle PQR is equilateral.

16. Two isosceles triangles ABC, DBC are on the same base BC. If AD is joined, show that the two triangles DAB, DAC are equal in all respects.

17. ABC is an isosceles triangle having AB = AC. The equal sides are produced beyond the base and the exterior angles bisected by lines which meet at O. Prove that BO = CO.

18. If the base of an isosceles triangle be produced, twice the exterior angle exceeds two right angles by the vertical angle.

19. ABC is a triangle with AC as its longest side. Find a point D in AC so that the angle ADB shall be twice the angle ACB.

20. ABC is a triangle with CB greater than CA. CD is cut off from CB equal to CA, and DE is drawn parallel to CA. Prove that DA bisects the angle CDE.

21. On a straight line AB as base construct isosceles triangles ABC, ABD on opposite sides of AB. Prove that CD when joined is a line perpendicular to AB. Let CD cut AB at E.

[Hint. Consider first Δs DAC and DBC, and, secondly, Δs CAE and CBE.]

* **22.** Two parallel lines AB, CD are cut by another line at X and Y : XP, XQ bisect the angles at X and meet CD at P and Q.

(*a*) State and indicate in the figure which angles are equal to AXP, and which equal to BXQ, and give your reasons.

(*b*) Prove that $YP = YX = YQ$.

(*c*) Prove that PXQ is a right angle.

* **23.** ABC is an isosceles triangle having $AB = AC$. The bisectors of the angles B and C meet in D. Prove that the triangle DBC is also isosceles; and prove that DA bisects the angle at A.

* **24.** Draw any line AB and bisect it at O. From O draw any line OC, making $OC = OA$. Prove $\angle OCB = \angle OBC$ and $\angle ACB = 1$ right angle.

* **25.** D is a point in the hypotenuse BC of a right-angled triangle ABC, and such that $\angle BAD = \angle ABD$. Prove that AD is half BC.

* **26.** An isosceles triangle ABC has each of the angles ABC, ACB double the angle BAC. The angle ABC is bisected by a line which meets AC in D. Show that the triangles ABD, DBC are isosceles, and that the angle ADB is three times as great as the angle BAC.

* **27.** ABC is a triangle : AD bisects BAC and CE is drawn parallel to DA, meeting BA produced at E ; prove that $AC = AE$.

* **28.** The angle BAC is bisected by the straight line AD. E is any point on AB, and a circle with E as centre and radius EA cuts AD in X. Show that XE is parallel to AC.

* **29.** ABC is an isosceles triangle. ($AB = AC$.) The angle ABC is bisected by BK, meeting AC at K. Through K is drawn KX, parallel to CB, cutting AB at X. Prove $BX = XK = KC$.

* **30.** ABC is an isosceles triangle in which $AB = AC$. BA is produced to X, and the angle XAC is bisected by the straight line AR. Prove that AR and BC must be parallel.

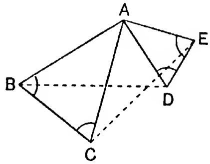
* **31.** ABC is an isosceles triangle with AB equal to AC. BA is produced to D, and through A a line AF is drawn parallel to BC. Prove that AF bisects $\angle DAC$.

[*N.B.*—This is a converse of the preceding. A direct proof is to be preferred to one involving *reductio ad absurdum.*]

* **32.** The bisector of the internal angle A of a triangle ABC meets BC in D. Through D a straight line DEF is drawn parallel to CA, meeting BA in E and the bisector of the external angle A at F. Prove that D, A, F are equidistant from E.

***33.** On the equal sides AB and AC of an isosceles triangle ABC are described externally equilateral triangles ABK and ACL. Prove CK = BL.

***34.** In the figure drawn ABC and ADE are two triangles, such that the angles marked at B, C, D and E are all equal.



Prove :

(i) That the angle BAC = the angle DAE.

(ii) That the angle BAD = the angle CAE.

(iii) That BD = CE.

***35.** One side DC of a rhombus ABCD is produced to E, and the exterior angle BCE bisected by a line CF. Prove that CF is parallel to the diagonal DB.

[A rhombus is a quadrilateral with all 4 sides equal.]

***36.** AB, CD are parallel lines cut by a third line at L and M respectively. Another line LN is drawn, cutting CD at N and such that the angle ALM = the angle BLN. Prove that the triangle LMN is isosceles.

***37.** If the angles at the base of an equilateral triangle be bisected by two lines which meet at a point within the triangle, the two lines drawn from this point parallel to the sides of the triangle divide the base into three equal parts.

***38.** ABC is a straight line which meets BD at B, and the angles ABD, CBD are bisected by BE, BF. If EF be drawn parallel to AC, prove that it is bisected by BD.

***39.** On a given base describe an isosceles triangle, if the sum of its height and one of its equal sides is also given.

***40.** ABC is a triangle right-angled at C, D is the mid-point of the hypotenuse AB; CM is perpendicular from C on AB.

Prove that the angle DCM is equal to the difference of the angles at A and B.

***41.** ABCD is a four-sided figure with AB parallel to CD; and the $\angle C = \angle D$; prove that AD = BC. [*N.B.*—AD and BC are *not* parallel.]

[Hint. Produce AD and BC to meet at K. Prove Δs KAB, KCD isosceles.]

* 42. ABC is a triangle. BM, CN are perpendiculars from B to AC and from C to AB respectively. If $BM = CN$, prove that the triangle is isosceles.

* 43. ABCD is a quadrilateral; AB and DC meet when produced in E. The angle BAD is equal to the angle CDA, and the angle ABC is equal to the angle BCD. Prove that AB is equal to CD.

* 44. The bisector of the angle A of a triangle ABC cuts BC at P; through P lines are drawn parallel to BA, CA, cutting CA, BA at Q and R. Prove that AQPR has all its sides equal.

* 45. AB and CD are two diameters of a circle, C and D are joined to B. Prove that CB and BD will bisect the angles made with AB by a straight line through B parallel to CD.

* * 46. XAB and YAB are two triangles standing on the same base AB and on the same side of it, and are such that $XA = YB$ and $XB = YA$. Show that if XB and YA cut at C, then the triangles ABC and CXY are both isosceles.

* * 47. ABCD is a square: PAB, QBC are equilateral triangles constructed on the outside of the square: prove that

(i) $PC = QD$;

(ii) the angles BPC, BCP, CQD, CDQ are all equal,

and state the size of each of these angles.

* * 48. ABC is a triangle, right-angled at C, and $CA = CB$. From P, any point in AB, PM and PN are drawn perpendicular to the other sides. Show that $PM + PN$ is a constant length.

* * 49. In the triangle ABC, AD and AE are drawn to cut BC in D and E, so that $\angle BAD = \angle ACB$ and $\angle CAE = \angle ABC$; prove that $\angle DAE$ is bisected by the perpendicular from A to BC.

* * 50. O is the middle point of a line AB which meets two parallel lines at A and B: any other line through O meets the parallel lines in C and D. Prove that ACBD has its opposite sides parallel.

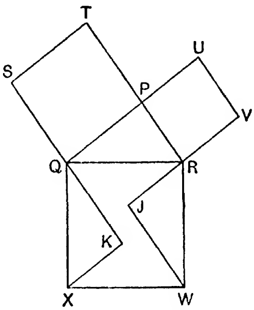
* * 51. Two isosceles triangles PAB, QAB have the same base AB, and Q lies within the triangle PAB. Prove that if AQ, BQ produced meet BP, AP respectively in X and Y, then $QX = QY$.

**52. ABC is an isosceles triangle whose base is BC; D is a point in AB, and E a point in AC produced such that DE is bisected at the point F in which it is cut by BC; prove that the sum of the lines AD, AE is equal to the sum of AB, AC.

[HINT. Produce FC to G making FG = FB; and prove △CGE is isosceles.]

**53. Angle QPR is a right angle. Squares are drawn on the sides of the triangle PQR. XK and WJ are perpendiculars drawn to SQ and VR produced. Prove that P, J and K are in the same straight line.

[HINT. Join PJ. Prove △s QPR, WJR congruent. Then prove ∠JPR = 45°. Similarly, etc.]



**54. ABC is an isosceles triangle having AB = AC. In AB produced and in AC respectively points D and E are taken so that BD = CE, and DE is joined. Prove that DE is bisected by BC.

**55. ABC is an isosceles triangle (AB = AC): BA is produced to D so that AD = AB. Prove that DC is perpendicular to BC.

**56. ABC is an isosceles triangle having AB = AC. D, E are points in AB, AC such that AD = AE. If CD, BE meet in F, prove that FB = FC.

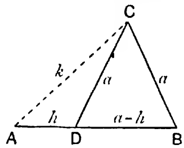
**57. ABCD is a square. On AD and BC, internally, are described equilateral triangles ADE, CFB. FC and DE meet at H, FB and AE at G; prove that FGEH is a rhombus.

**58. ABC is an isosceles triangle having AB = AC; from B and C straight lines are drawn at right angles to BC, meeting CA and BA produced in D and E respectively; prove that BD = CE.

**59. ABC is an equilateral triangle. DBC is an isosceles triangle on the same base BC and on the same side of it. If ∠BDC = $\frac{1}{2}$∠BAC, prove AD = BC.

* * **60.** Through D the mid-point of the side AB of a triangle ABC, DE is drawn parallel to BC, meeting the bisector of the angle ABC at E. Prove AE perpendicular to BE.

* * **61.** The figure shows three equal bars AB, BC, CD jointed at B and C. The three are placed on a table, and the bar AB is kept fixed while the point D is gradually moved along AB from A to B, the joint C moving in consequence across the table. Prove that if the straight line AC is drawn, then, in all positions of D, the triangle ADC has one of its angles double of another.



CHAPTER X.

INEQUALITIES.

§ **1.** Many propositions in Geometry are concerned in *proving* things equal as a necessary logical sequel to certain other facts *given* equal. Some propositions concerned with unequals are also important; and a group of these, known as **Inequalities**, is considered in this chapter.

We are familiar with the fact that it is shorter to walk from one point to another direct along the one straight line joining them, rather than by a circuitous route; and it is almost proverbial that the walking along two sides of a triangle is longer than walking direct. This is expressed geometrically by the proposition *Two sides of a triangle must be greater than the third.* (A formal proof of this proposition is not given here, but its truth is to be admitted whenever necessary.)

Another proposition, *If a side of a triangle is produced the exterior angle so formed must be greater than either of the two interior and opposite angles* is frequently useful, and is easily established when we remember what was proved in the first part of Proposition **1.**

The sign $>$ means " is greater than " or " are greater than."

" $<$ " " is less than " or " are less than."

" $\ngtr$ " " is not greater than " or " are not greater than."

" $\nless$ " " is not less than " or " are not less than."

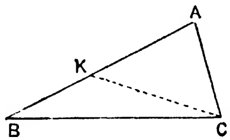
Strictly, all these signs are verbs and not adjectives. They are used for any tense of the verb.

PROPOSITION 6.

§ 2. **General Enunciation.** *In any triangle, if one side be greater than another, the angle opposite the greater side must be greater than the angle opposite the other.*

Particular Enunciation. ABC is the triangle, with AB given greater than AC, to prove $\angle$ ACB is greater than $\angle$ B.

Construction. From AB cut off AK equal to AC, Join KC.



Proof. $\angle$ AKC = $\angle$ ACK (isos.),

but $\angle$ AKC is greater than $\angle$ B (ext. $\angle$ of $\triangle$BKC is greater than int. opp. $\angle$);

$\therefore$ $\angle$ ACK is greater than $\angle$ B,

but $\angle$ ACK is only a part of $\angle$ ACB;

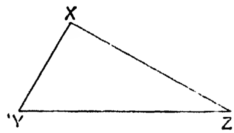
$\therefore$ $\angle$ ACB must be greater than $\angle$ B. **Q.E.D.**

§ 3. The following proposition (proved by *Reductio ad absurdum*) is the converse of the preceding:

PROPOSITION 7.

General Enunciation. *In any triangle, if one angle be greater than another, the side opposite the greater must be greater than the side opposite the other.*

Particular Enunciation. XYZ is the triangle, with $\angle$Y greater than $\angle$Z, to prove XZ must be greater than XY.



Proof. If XZ is not greater than XY, then

either (i) XZ = XY or (ii) XZ is less than XY;

(i) If XZ = XY the triangle is isosceles, and $\angle$Y = $\angle$Z, but this is contrary to what is given (so XZ is not equal to XY).

(ii) If XZ is less than XY, then by the preceding proposition $\angle$Y is less than $\angle$Z, again contrary to what is given (so XZ is not less than XY).

From (i) and (ii) we infer XZ must be greater than XY.

Q.E.D.

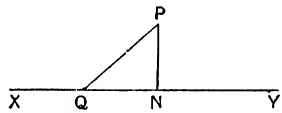
§ 4. It is almost proverbial that the perpendicular distance of a point from a straight line is the shortest of distances of that point to any point on the straight line; and by the distance of a point from a straight line is always meant the perpendicular distance. A formal proof, that the perpendicular is the shortest, is given here.

PROPOSITION 8.

General Enunciation. *Of all straight lines that can be drawn to a given straight line from a point outside it, the perpendicular is the shortest.*

Particular Enunciation. XY is the given straight line, and P the given point. PN is perpendicular to XY. To prove that PN is the shortest of all lines from P to XY.

Construction. Let Q be any point (not N) on XY. Join PQ.



Proof. The 3 angles of the $\triangle$PQN make up 2 right angles,

but $\angle$PNQ is a right angle,

and hence $\angle$PQN is less than a right angle;

$\therefore$ PN is less than PQ.

Hence the distance PN is less than the distance of P to any other point on XY;

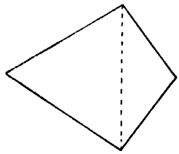
$\therefore$ PN is the shortest of all lines from P to XY.

Q.E.D.

EXAMPLES 10 (RIDERS).

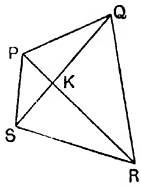
Two sides of a triangle greater than the third.

1. Prove that the perimeter of a quadrilateral is greater than the sum of its diagonals.



[Hint. Prove the perimeter is greater than twice one diagonal, and then greater than twice the other diagonal, and add the results.]

2. Prove that the perimeter of a quadrilateral is less than twice the sum of its diagonals.



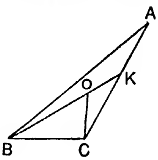
[Hint. $PK + KS > ?$ etc., etc., etc., and add.]

3. The sum of the distances of any point from the vertices of a triangle is greater than half the perimeter of the triangle.

[Hint. Join the point to the 3 corners.]

Exterior angle of a triangle greater than interior opposite angle.

4. O is any point within a triangle ABC. Prove that the angle BOC is greater than the angle BAC.

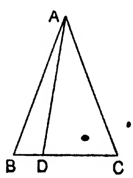


[Hint. Produce BO to cut AC in K.]

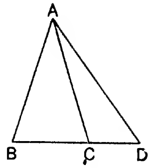
5. If two triangles stand on the same base and have equal vertical angles, prove that the vertex of each triangle must lie outside the other triangle.

Greater side opposite the greater angle and the converse.

6. Prove that a straight line joining the vertex of an isosceles triangle to a point on the base (not produced) must be less than one of the equal sides.



7. ABC is an isosceles triangle in which AB = AC. If BC is produced to D, prove that AD is greater than AC.



Miscellaneous.

8. Prove that the difference between any two sides of a triangle is less than the third side.

9. ABC is a triangle, C a right angle and D any point in BC. Prove that AB is greater than AD, and that each of these is greater than AC.

10. ABC is an isosceles triangle (AB = AC), D any point in BC the base; show that the angles ADB and ADC are each greater than the equal angles of the triangle.

*11. ABC is a triangle and ACKL a quadrilateral on the same base AC and on the same side of it. The triangle is entirely within the quadrilateral. Prove that the perimeter of the triangle is less than the perimeter of the quadrilateral.

*12. ABC is a triangle, D is the middle point of BC. Show that the perimeter of the triangle ABC is greater than twice AD.

*13. ABC is a triangle and the sides BA and CA are produced to any points Y and Z. A point K is taken within the angle YAZ. BK and KC are joined. Prove that the angle K is less than the angle BAC.

*14. ABCD is any quadrilateral. Prove that the sum of the diagonals AC and BD is greater than the sum of either pair of opposite sides.

*15. If O is the point within the triangle ABC, show that the sum of OA, OB, OC is greater than half the sum of BC, CA, AB.

*16. ABC is an equilateral triangle and P any point within it. Prove that a triangle can always be constructed with sides equal respectively to PA, PB, PC.

*17. ABC is an equilateral triangle. D is any point in BC. E is the middle point of AD. Prove that AE is less than CE.

*18. A and B are any two fixed points on the same side of a straight line XY, which is the perpendicular bisector of a given straight line AA′. BA′ cuts XY at P. Show that the sum of the distances of A and B from P is less than the sum of the distances of A and B from any other point on the line XY.

*19. XY is a straight line and P a point outside it. PN is perpendicular to XY. Q and R are any two points on XY on same side of N. If QN > RN, prove that PQ > PR.

*20. The diagonals of a square ABCD intersect in E, and F is any point within the square. Prove that EA + EB + EC + ED is less than FA + FB + FC + FD.

[HINT. Consider $\triangle$FAC, then $\triangle$FBD.]

*21. ABC is any triangle and D any point inside it. On BD and DC respectively are taken any points K and L. Prove that the angle BKL is greater than the angle BAC.

*22. In a triangle ABC, O is the middle point of the side BC; prove that AB + AC > 2AO.

[HINT. Produce AO to F, making OF = AO.]

*23. ABC is a triangle, its side BC is produced to D, BM is the bisector of the angle ABC, CN the bisector of the angle ACD.

Prove that BM and CN can never be parallel.

*24. The triangle ABC has AB greater than AC. AD bisects the angle BAC and meets BC in D. Prove that the angle BDA is obtuse.

*25. ABC is a triangle, right-angled at B; BD is drawn perpendicular to AC, and produced both ways. Find for what positions of any point P on this line the angle APC is obtuse and for what positions it is acute.

*26. If P be any point within a triangle ABC, prove that BC + CA + AB is greater than PA + PB + PC.

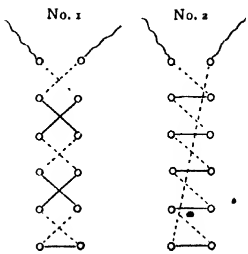
**27. PQR is a triangle having PQ greater than PR; and M is the middle point of QR. Prove that angle PMQ is obtuse.

**28. P is any point in the side BC of a triangle ABC, PM and PN are drawn perpendicular to AB and AC; show that BC is greater than MN.

**29. The median of a triangle RST drawn through R is greater than the bisector of the angle R.

**30. ABC is a triangle, and D is a point in BC. Prove that one at least of the lines AB, AC is greater than AD.

**31. Two ways of lacing a shoe are indicated. Which way is the more economical of lace?

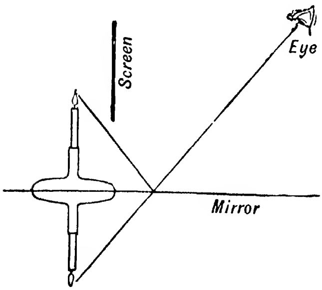


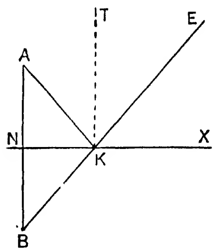
CHAPTER XI.

IMAGES.

(This chapter should be omitted entirely in a first reading of the subject.)

* § 1. If a lighted candle in a candlestick is standing on a plane mirror, as is indicated in the first diagram, and if a screen is interposed, the eye can only see the light by reflection. The essential features are shown in the second diagram, and in addition the line TK, perpendicular to NX, is drawn.





B is called the image of A in NX. The eye E looks towards B and the reflection is at K.

When light is reflected by a mirror, the angle of reflection is equal to the angle of incidence.

Here AKT is the angle of incidence and EKT the angle of reflection.

Draw AN perpendicular to NX, and produce it to cut EK produced at B.

Now, since TK is perpendicular to NX,

$$\angle AKN = \angle EKX,$$

$$\text{but} \quad \angle EKX = \angle BKN \text{ (vert. opp.)};$$

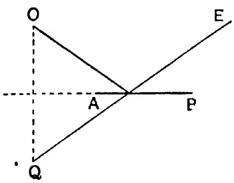
$$\therefore \quad \angle AKN = \angle BKN;$$

and remembering the angles at N are right angles, it is easy to prove $\triangle$s AKN and BKN congruent, and, in particular, $AN = BN$.

Hence, the easy construction for obtaining the image of a point is, "Draw a perpendicular and double it." (If one stands in front of a looking-glass one's image seems as far behind the glass as one is in front of the glass.)

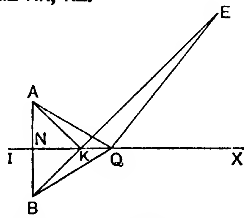
It is most important to note that the position of the image of A is quite independent of the position of E. For instance, the position of the image of some lamp, in a mirror, is independent of the position of the observer. In the special case, when one is looking at one's own image, it should be noticed that if one moves, one's image moves precisely the same amount.

Also note that it is quite possible to see an image of an object even though the mirror does not extend so far as to be opposite the object. E will see the image of O at Q, even

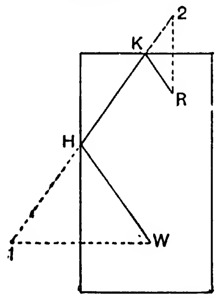


though the mirror is as short as AB. Recollect that you may see, in a looking-glass over a mantelpiece, images of objects quite at the side of the room.

* § 2. It is noticeable that **light**, from A to E, reflected by the plane mirror IX, **travels by the shortest path.** For suppose that the light were reflected to the eye at any other point Q. The path would be AQ, QE, and that is the same length as BQ + QE. Now, if the light were reflected at K, the path would be AK, KE, and that is the same length as BK + KE or BE. But the two sides BQ + QE of the triangle are greater than the third side BE, so that *any* path is longer than BK + KE, *i.e.* than AK, KE.

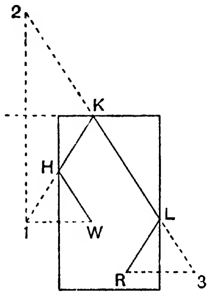


* § 3. When the angle of incidence is equal to the angle of reflection, "images" are always obtained by drawing perpendiculars and producing them their own length. If the cushions of a **billiard table** are perfectly elastic, and if the cue ball is struck quite centrally, this law obtains. To get from

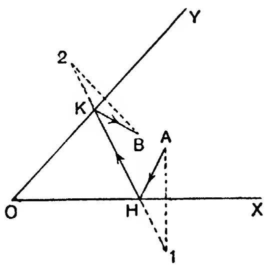


W to R off two cushions, as is indicated, first obtain 1 (the image of W in the cushion off which the reflection is to take place); secondly, obtain 2 rather similarly. The join of 1, 2 gives the points H and K, and the course WHKR follows.

For three reflections obtain 1 and 3 (these images are clear); and then 2, which is the image of 1 in the cushion produced (*i.e.* the image of the image). The join of 2, 3 gives K and L. The join of K, 1 gives H; and the course WHKLR follows.

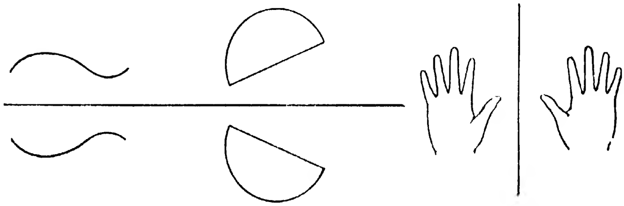


* § 4. If light has to get from A to B, after reflection once in each of the plane mirrors OX and OY, the course is AHKB, and its construction is easy if we obtain the images 1 and 2 first.

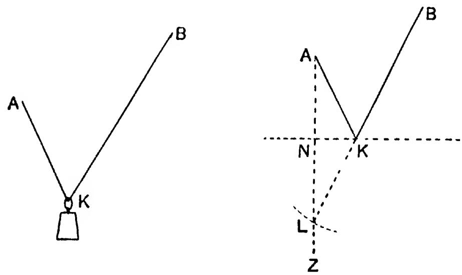


[In the instrument called the **sextant** there is reflection once in each of two mirrors inclined at an angle.]

* § 5. The image of a line, or indeed of any figure, is simply the aggregate of the images of all the points composing that line or figure, and the fact that the image implies a perpendicular bisector only shows that **symmetry** and **images** often produce the same problems. We might consider one glove to be the image of the other, in the straight line, or that the whole figure is symmetrical with regard to the axis of symmetry shown.



* § 6. **A weight hangs** from a ring which can slide **freely** along a cord fastened to A and B. As a matter of fact, the tension of the cord will be the same throughout, and the two portions, AK and BK, will be equally inclined to the vertical (for light



we should say the angle of incidence is equal to the angle of reflection). To construct the position of the cord, when its length and terminal points are known; with centre B and radius equal to the length of the cord obtain L on the vertical AZ. The perpendicular bisector of AL gives the point K.

EXAMPLES 11 (MOSTLY GEOMETRICAL DRAWING).

* 1. Prick off the points A, E and the extremities of the line OY (last page). Draw a diagram to show the course of a ray of light, from A to E, with reflection in a mirror represented by OY. Measure both the direct course and the reflected course in cm.

* 2. Prick off the points A, F and the extremities of the line OX (last page). Draw a diagram to show the course of a ray of light, from A to F, with a reflection in OX. Measure the direct and reflected courses in cm.

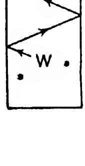
* 3. Prick off the points Q and E and the extremities of the line OY (last page). OY is an ivy-clad wall. A competitor in a race starts from Q, has to pluck a leaf of ivy from the wall and finish at E. Draw a diagram of the shortest course. If the scale is 100 yds. to 6 inches, what is the length of the shortest course?

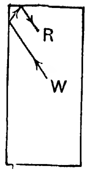
* 4. Two villages, A and B, are 2 miles apart. A is one mile direct from a straight road XY, and B is $1\frac{1}{2}$ miles direct from the same road. Draw a diagram to show the shortest route from A to B if the road has to be touched somewhere. Give the length in miles of the shortest route.

* 5. Prick off the points S and D and the extremities of the lines OX, OY (last page). Show the course of a ray of light from D to S with reflections in OX and OY. What is the total length in inches?

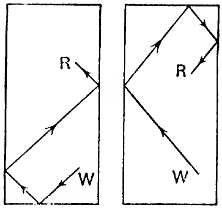
* 6. If the points E and T (last page) are billiard balls, and the straight lines OX and OY are cushions, prick off the necessary points, and show, by an accurate diagram, the course of the ball E to the ball T off the cushions OX and OY. Measure the length of the course of E if the diagram is on the scale 1 in. to 1 ft.

* 7. Draw a rectangle whose sides are 12 ft. and 6 ft. [1 cm. to 1 ft. is a convenient scale.] Take 2 convenient points W and R on it. Suppose the rectangle represents a billiard table, draw accurately the course from W to R off two cushions as indicated.





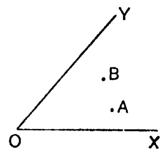
* **8.** Suppose the course were as indicated, construct that accurately.



* **9.** With the same rectangle show the course from W to R in the two ways indicated.

* **10.** ABCD is a billiard table and X a given point on it: obtain a construction, giving the path of a ball which is hit from X on to AB, thence rebounds on to BC, and again rebounds so as to fall into the pocket at D. [AB = 6 ft., BC = 12 ft.]

* **11.** Draw a diagram to show the course from the white ball (W) to the red ball (R) on a billiard table, if W is reflected by each of the four cushions in order. Take W and R in convenient positions.

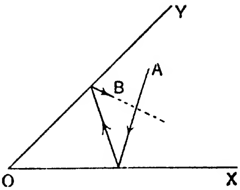


* **12.** OX and OY represent two mirrors. Construct the course of a ray of light, from A to B, reflected twice, once by each of the mirrors. Take A and B conveniently.

* **13.** Prick off the points E, C and the extremities of the line OY (last page). A cord, 7 cm. long, is attached to the points E and C. A weight slides freely on the cord. If OY is a vertical line, draw an accurate diagram of the position.

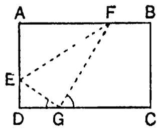
** **14.** Prick off the points T, P, Q, F, C, A (last page). Rule in the straight lines FC and TP. Suppose FC and TP represent mirrors. Construct the course of a ray of light from A to Q after reflection in each of the two mirrors. What is the length of the course?

**15. Light goes from A to B by double reflection in the mirrors OX and OY as indicated. Prove that the angle between the initial and final directions of the light is twice the angle between the mirrors.



**16. The rectangle ABCD represents a billiard table (AB is a long cushion). Take on it two points W and R in convenient positions. Construct the course from W to R to show 6 reflections in the cushions AB, CD, BC, AB (again), CD (again), AD in this order.

**17. ABCD is a rectangle (not drawn to scale). AB is 8·1 inches, BC is 5·2 inches. The corner A is folded over on to CD in such a manner that the angles EGD and FGC are equal (see diagram). Describe the geometrical construction necessary to draw the figure to scale. Draw an accurate figure (1 cm. to 1 inch), and measure the length of the crease EF.

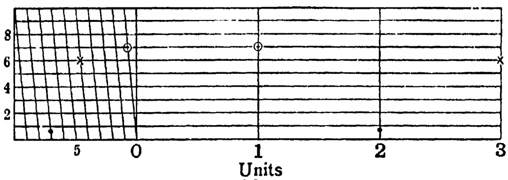


CHAPTER XII.

SMALL MEASUREMENTS.

(This chapter should be omitted entirely in a first reading.)

* § **1.** The length of a line may be judged, by eye, to within a hundredth of an inch, with the use of a ruler showing inches and tenths only. We can go still further if we open a pair of dividers to the length we wish to measure, and use them on the **diagonal scale.** [Dividers adjustable to a nicety with a screw, and furnished with needle points too, are convenient for good work.]



The bottom line merely shows units and tenths of a unit. The diagonals steadily lean to the left, and in 11 lines (*i.e.* 10 spaces) the total lean is one-tenth of a unit (this is seen easily as the top is divided into tenths too), so that the lean for each space is $\frac{1}{10}$ of $\frac{1}{10}$ of a unit or $\frac{1}{100}$ of a unit.

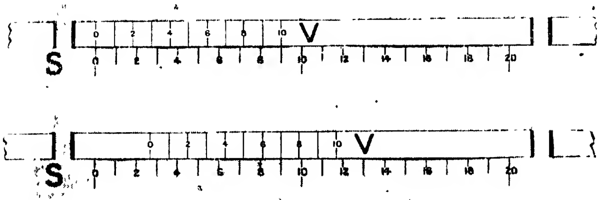
From X to X is 3·46 of a unit.

From ⊙ to ⊙ is 1·07 of a unit.

From . to . is 2·706 of a unit (the third place of decimals being "hit off" by eye).

A diagonal scale is not much good without a pair of dividers.

* § 2. **Verniers** (Pierre Vernier, *circa* 1600, invented the device) are a useful contrivance on some scientific instruments (not on G.D. instruments, but on callipers, barometers, sextants, theodolites, etc.) for measuring to a considerable degree of accuracy. Subdividing a small distance (like a millimetre) into ten equal parts would be extremely difficult, probably inaccurate, hopeless to read with the naked eye and hard enough with a microscope. Now it is almost as easy to divide 9 mm. into ten equal parts as it is to divide 10 mm. into the same number of parts, and it is using this fact which makes the construction of a vernier possible. The accompanying divisions are of course big, and perhaps a vernier was unnecessary for them, but you will see the principle better from a big figure.



First there is a scale, S, divided ordinarily, we will say into units (they may be inches, or cm., or mm., or degrees, or anything else).

Secondly, there is the V scale (the "Vernier" scale).

Consult the upper pair of scales in the figure. Clearly 10 V divisions are equal to 9 S divisions. That means each V division is $\frac{9}{10}$ of an S division, or that the difference in the length of the V and S divisions is $\frac{1}{10}$ of an S division. *In other words, the vernier reads to tenths.*

The V scale is made to move along the S scale, by sliding

or by a screw motion (Verniers have nothing whatever to do with "Slide-Rules").

Now consider the lower pair of scales in the figure. The S nought to the V nought is to be read. The S scale tells the distance as 2 and a bit; the V scale (as there is a "coincidence" at 7) tells us that the bit is $\frac{7}{10}$.

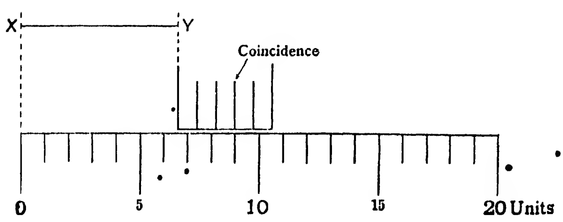
So that, in all, the distance is $2\frac{7}{10}$ *units.*

[The following is only added, for your convenience, if you do not see why the "coincidence" at the 7 on the V scale means $\frac{7}{10}$ (it does not matter what the number on the S scale is).

From S2 to S9 = 7 × *ten* tenths,
and V nought to V7 = 7 × *nine* tenths;
but S9 and V7 coincide;
∴ (by subtraction) S2 to V nought = 7 × *one* tenth
$= \frac{7}{10}$.]

In this figure the divisions are *large* to make the principle clear so that apparently there might not be a "coincidence," and you naturally question what happens then. In practice there always *is* a "coincidence," because the divisions are properly small. It is generally a question of which we shall choose of 2 or 3, and we must choose the best, or split the difference when it is a question of 2, *e.g.* if there seems to be a "coincidence" both at the 3 and at the 4, say that $3\frac{1}{2}$ is correct.

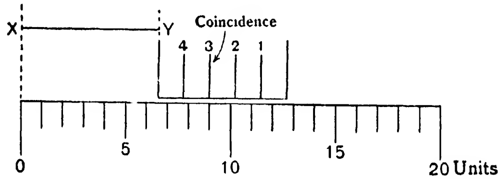
Of course verniers need not read tenths. Below is one to read fifths and XY $= 6\frac{3}{5}$ units.



If a scale were divided into units, then a vernier to read $\frac{1}{n}$th of a unit would be **n − 1 units long divided into n equal parts.**

In normal British barometers the scale is divided into twentieths of an inch. A vernier 24 of these long, divided into 25 equal parts, reads to $\frac{1}{25}$ of $\frac{1}{20}$ inch, *i.e.* to $\frac{1}{500}$ inch.

* * § **3.** There is another kind of vernier scale also, to read $n$ths; it is **n + 1 units long divided into n equal parts.** It is used on the same principle as the preceding, but here each division on the V scale is *greater* than each division on the S scale by $\frac{1}{n}$ of a unit, and in consequence it has to be numbered in the reverse way.



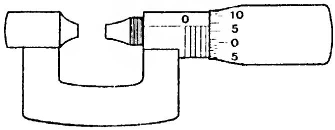
The distance XY = $6\frac{3}{5}$ units.

(6 from the S scale and $\frac{3}{5}$ from the V scale, because there is a coincidence at the 3 of the V scale. Note that the numbers on the V scale read backwards.)

* * § **4.** With **screw-gauges** (or Micrometer Callipers) one can read diameters (like the thickness of wire) very accurately. It is much better to get accustomed to a particular instrument (there are many patterns) practically, than only to read a book description. The principle is to take into account fractions of a turn of a screw. If the screw advances $\frac{1}{40}$ inch for a complete turn (*i.e.* the "pitch" of the screw is $\frac{1}{40}$ inch); then, if there are 25 divisions on the circular part, one division would mean $\frac{1}{25}$ of $\frac{1}{40}$ in. or $\frac{1}{1000}$ in. It may be necessary to take into account "zero readings," especially if the instrument is much worn.

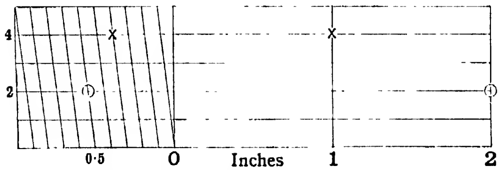
[" Zero Readings." Very few instruments are perfectly graduated, and they often do not read "nought" when they ought to. Thus, when the ends are in contact (not screwed tight to wear the instrument badly), take the reading, which is called the "Zero Reading," and allow for it in subsequent calculations.]

The figure shows a screw-gauge. The pitch of its screw happens to be $\frac{1}{2}$ mm. There are fifty divisions on the screw-head. The opening is 4 millimetres and a bit. Closer inspection shows that the distance is somewhere near $4\frac{1}{2}$ (or 4·5), and then, more accurately, we note that the 6 tells the precise turn, so that the reading is 4·56 mm.

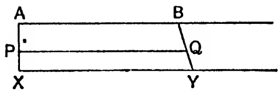


EXAMPLES 12 (GENERAL).

* 1. To what fraction of an inch does this diagonal scale read? What are the distances ⊙ ⊙ and XX ?



* 2. In the figure AB is 12 fifths of an inch and XY is 13 fifths of an inch. AP is $\frac{5}{8}$ of AX. How long is PQ? [Give your answers both in fifths of an inch and also in full inches.]

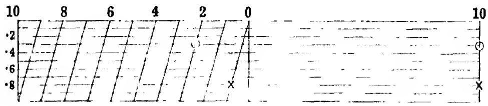


* 3. A scale on a map shows degrees and sixths of a degree. It is necessary to be able to read single minutes accurately, and

the diagonal scale method is adopted. How many spaces between the parallels must there be? How many parallels in all are there, excluding the sloping lines? Sketch a diagram (on no particular scale) to show the appearance of the diagonal scale, and by two crosses show 3° 45′.

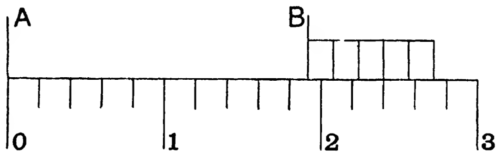
*** 4.** What do the distances ⊙ ⊙ and X X represent? If the scale is 1 : 430,000, what should be these distances on paper?

[*N.B.*—It is extremely improbable that 1 : 430,000 is quite correct for the scale shown, so do not wonder if your check by measurement does not quite agree with your calculation.]



The Figure represents a Scale of Miles.

*** 5.** The figure shows a vernier used with a scale of inches. What is the length of each of the divisions of the vernier, and what is the length AB shown by its position in the figure? Draw a rough figure to show the position of the vernier when denoting $1\frac{7}{25}''$. (Assume that each division of the scale is a fifth of an inch.)



*** 6.** Supposing that the rulings on your paper correctly show half-centimetres, construct a vernier to read to the fifth part of a half cm. Show a scale with a vernier in position to read $12\frac{3}{5}$ half cm.

*** 7.** On the squared paper draw a line and number it, below the line, as a scale up to 30. [0, 10, 20, 30 units.] Separately make a vernier to read fifths of a unit. Now draw the vernier in position to read $12\frac{3}{5}$ units. Mark the "coincidence" clearly.

* **8.** Draw a straight line 30 units long, and number it (0, 5, 10, 15, 20, 25, 30) at intervals below the line, to represent a scale of units.

Construct a vernier which can be used to read sixths of a unit.

Now copy the vernier above the line of the scale in such a position as to show $17\frac{5}{6}$ units.

Also obtain $17\frac{5}{6}$ units by the diagonal scale method and use this as a check for your former work.

* **9.** *Use the paper broadways for this question.*

Draw a straight line 40 units long and number it [0, 5, 10, ... 40] at intervals, below the line, to represent a scale of units.

Construct a vernier which can be used to read sevenths of a unit.

Now copy the vernier above the line of the scale in such a position as to show $23\frac{4}{7}$ units.

Also obtain $23\frac{4}{7}$ units by the diagonal scale method [and use this as a check for your former work].

[It will depend on the width of the rulings on your paper if the numbers are appropriate. Make your line 20 units long if more convenient, and show $3\frac{4}{7}$ units.]

* **10.** A theodolite is graduated to read single degrees. A vernier has to be made for it to read minutes. Describe, in words, the vernier.

* **11.** A theodolite is graduated to read half-degrees, and a vernier has to be made for it to read minutes. Describe, in words, the vernier.

* **12. Construction of a Model Vernier.**

(See the figure on page 132.)

(*a*) Rule a straight line on a piece of plain paper, making it go right across, excepting a margin of about an inch at each end.

(*b*) Use a separate piece of ruled paper to mark on it, throughout its entire length, divisions at equal intervals.

(*c*) At each of these marks *rule* short vertical lines *below* the line and number them neatly 0, 1, 2, ... , etc.

(*d*) Now, on the ruled paper, draw a straight line perpendicular to the rulings and make it *nine* divisions long.

(*e*) Divide this carefully into *ten* equal parts.

(*f*) Take a long narrow strip of plain paper and mark on it, more or less centrally, a scale equivalent to that produced in (*e*); and at each of the marks *rule* short vertical lines. Call this the vernier scale.

(*g*) By threading the narrow strip through a couple of slots at each end of the original scale a sliding vernier to read tenths of a division is constructed.

* **13.** On squared paper draw a scale of centimetres. Make a movable vernier 4 cm. long and divided into 5 equal parts. With the vernier, to what degree of accuracy can you measure with your scale ?

* **14.** On squared paper draw a scale of centimetres. Divide a length of 7 cm. up into eight equal parts and mount it to slide along your scale as a vernier. To what fraction of a centimetre does your vernier read ?

** **15.** A theodolite is graduated to show degrees, and each degree is subdivided into 3 equal parts. How many minutes go to one part ? A vernier, 59 of those parts long, slides along the scale of degrees, and it is divided up into 60 equal parts. To what degree of accuracy does the theodolite read ?

** **16.** There is a scale on a sextant showing single degrees, and each degree is subdivided into six equal parts. How many minutes go to one part ? A vernier, 59 of those parts long, is attached. To what degree of accuracy can the instrument be read ?

** **17.** A barometer is to be read to the $\frac{1}{500}$ of an inch. The scale shows twentieths of an inch. Into how many parts must the vernier be divided ? How long must the vernier be ?

** **18.** There is a scale of units, and two separate verniers are to be made to read tenths of units. One is 9 units long and the other is 11 units long. Into how many parts are the verniers divided ? They show $12\frac{3}{10}$ units. At what number on the main scale (not on the vernier scale) does the coincidence occur in the two cases ?

** **19.** Describe the screw-gauge and draw a careful picture of the instrument. If the pitch of the screw were $\frac{1}{50}$th inch, and if there were 20 divisions on the screwhead, to what degree of accuracy would the instrument read ?

CHAPTER XIII.

QUADRILATERALS.

§ 1. It must not be forgotten that in this book we are concerned with **plane figures** (*i.e.* flat figures). A **quadrilateral** is a plane figure bounded by four straight lines. Some quadrilaterals are sufficiently important to have names of their own also.

Definitions. A **parallelogram** is a four-sided figure whose opposite sides are parallel.

[See the figures overleaf.]

A **rectangle** is a parallelogram which has *one* of its angles a right angle.

(It is easy to prove that if *one* angle of a parallelogram is a right angle *all* its angles must be right angles; and one always thinks of a rectangle as having all its angles right angles.)

A **rhombus** is a figure which is contained by four equal straight lines.

(A rhombus is often called a **diamond**, especially when it is point uppermost.)

(The chief thing to prove about a rhombus is that its diagonals must bisect each other at right angles.)

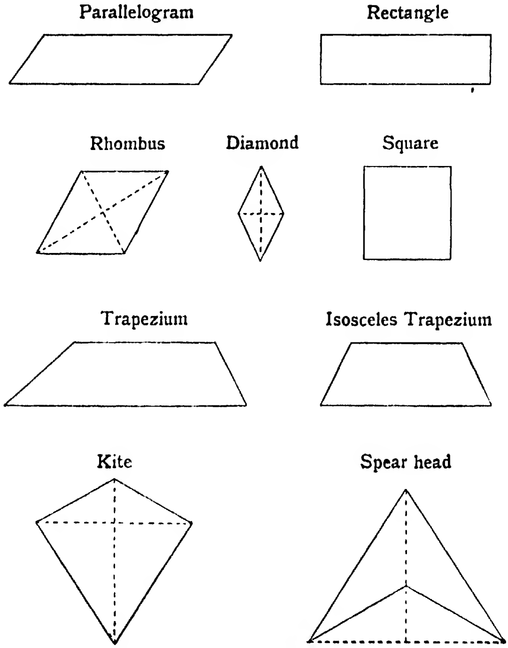
Do not learn any definition of a **square**, but assume obvious facts.

A quadrilateral which has one pair of opposite sides parallel is called a **trapezium.**

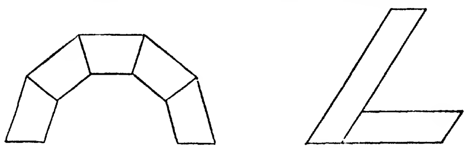
(A trapezium which has its non-parallel sides equal is called an **isosceles trapezium.**)

A **kite** is a quadrilateral with one pair of adjacent sides equal, and the other pair also equal (not to the first pair). Its diagonals

are at right angles. (If this quadrilateral has a re-entrant angle it becomes spear-headed.)



Many familiar objects are in the shape of special quadrilaterals; thus picture frames are often rectangular, the ace of



diamonds is a rhombus, pocket-handkerchiefs are usually square, the stones of an arch might be trapezia, and it is possible to make

the letter L out of two equiangular parallelograms, while the very name kite suggests the shape of an object that we know well.

One must remember that anything proved about a parallelogram is bound to be true also of a rectangle (for the definition of a rectangle says it is a parallelogram); it is assumed that a square is a parallelogram; it is easy to prove that a rhombus must be a parallelogram; so that, as the name parallelogram includes rectangles, squares and rhombi, anything we can prove about the parallelogram must be true (and needs no further proof) for all these figures.

Hence the proposition which follows is comprehensive.

PROPOSITION 9.

§ 2. **General Enunciation.** (i) *The opposite sides and angles of a parallelogram must be equal,* (ii) *the diagonals must bisect the parallelogram, and* (iii) *the diagonals must bisect each other.*

Particular Enunciation. WXYZ is the parallelogram.

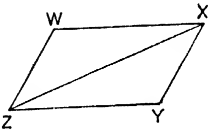
To prove (i) WX = YZ and ZW = XY,

$\angle W = \angle Y$ and $\angle WXY = \angle YZW$;

(ii) the diagonal ZX bisects WXYZ, and similarly WY.

(iii) (this is given separately later).

For (i) and (ii). **Construction.** Let ZX be one diagonal (perhaps this is already drawn).



Proof. In the triangles WXZ and YZX,

$$\because \begin{cases} ZX \text{ common,} \\ \angle WXZ = \angle YZX \text{ (alt.),} \\ \angle XZW = \angle ZXY \text{ (alt.),} \end{cases}$$

∴ the triangles WXZ and YZX are congruent, and, in particular,

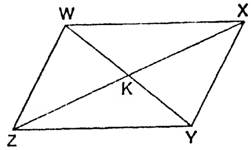
$$\begin{cases} WX = YZ, \\ ZW = XY, \\ \angle W = \angle Y, \\ \text{and } \triangle WXZ = \triangle YZX \text{ in area, so that the diagonal } ZX \text{ bisects } WXYZ \text{;} \end{cases}$$

similarly, by drawing the diagonal WY, we can prove

$$\begin{cases} \angle WXY = \angle YZW, \\ \triangle WXY = \triangle YZW \text{ in area, so that the diagonal } WY \text{ also bisects } WXYZ. \end{cases}$$

For (iii). **Construction.** Join WY and let the diagonals cut at K.

Particular Enunciation. To prove WK = KY and XK = KZ.



Proof. In the triangles WKX and YKZ,

$$\because \begin{cases} WX = YZ \text{ (proved above)}, \\ \angle WXK = \angle YZK \text{ (alt.)}, \\ \angle KWX = \angle KYZ \text{ (alt.)}, \end{cases}$$

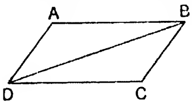
$\therefore$ the triangles WKX and YKZ are congruent, and, in particular,

$$\begin{cases} WK = KY, \\ XK = KZ. \end{cases}$$ **Q.E.D.**

§ 3. The following fact is frequently useful :

Straight lines which join the extremities of equal and parallel straight lines, towards the same parts, must themselves be equal and parallel.

The proof is simple, and only an outline is given here.*



We are given AB and DC are equal and parallel, and have to show AD and BC must be equal and parallel (in other words, ABCD must be a parallelogram).

The construction is, Join BD, and in the proof we consider the congruent triangles ABD and CDB.

This result may be assumed, when necessary, in any riders.

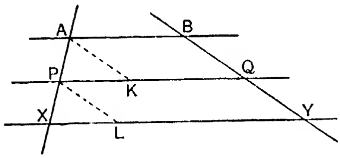
* The full proposition is given in the Appendix, Proposition A, page 225.

PROPOSITION 10.

§ **4. General Enunciation.** *If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them (shortly called transversal) are equal, then the corresponding intercepts on any other straight line that cuts them (another transversal) are also equal.*

Particular Enunciation. AB, PQ, XY are the three parallels, cut by the transversals APX and BQY. It is given that AP = PX, to prove BQ = QY.

Construction. Draw AK and PL parallel to BQY.



Proof. In the $\triangle$s APK, PXL,

$$\because \begin{cases} \angle PAK = \angle XPL \text{ (corresponding ; AK, PL being parallel)}, \\ \angle APK = \angle PXL \text{ (corresponding ; PQ, XY being parallel)}, \\ AP = PX \text{ (given)}, \end{cases}$$

$\therefore$ $\triangle APK \equiv \triangle PXL$,

and, in particular, AK = PL.

Now AK = BQ (opposite sides of a parallelogram),

and PL = QY (for the same reason).

$\therefore$ BQ = QY. **Q.E.D.**

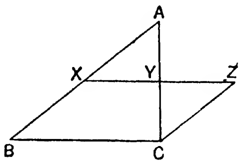
(Similarly, if there are more than three parallels.)

[*N.B.*—BQ is not equal to AP.]

§ 5. The following proposition is not strictly about quadrilaterals, but a property of parallelograms is used in it, and so an outline of the proof is given here.*

The join of the mid-points of two sides of a triangle must be parallel to the base and half thereof.

(This is a special case of a more comprehensive proposition in Vol. II., given later, but sufficiently important to be considered here.)



ABC is the Δ.

X and Y are the mid-points of AB and AC.

XY is produced to Z so that $YZ = XY$.

ΔCZY and ΔAXY congruent,

and hence $ZC = XA$ and $\angle Z = \angle AXY$; hence ZC and AX are $\parallel$;

so that ZC and XB are equal and parallel.

Hence XZCB is a parallelogram, etc.

* The full proposition is given in the Appendix, Proposition B, page 226.

* * § 6. A **Median** of a triangle is a straight line joining a vertex to the middle point of the opposite side.

The medians of a triangle are concurrent and trisect each other.

ABC is the triangle.

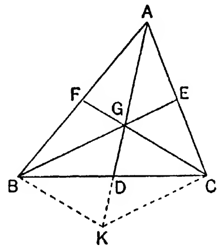
E and F are the mid-points of *two* sides.

BE and CF meet at G.

AG is joined and *doubled* by producing to K.

BK and CK are joined.

GK and BC meet at D.



*Outline of proof.** F and G are mid-points of two sides of $\triangle$ABK;

$\therefore$ FG (*i.e.* FC) is parallel to BK.

Similarly, BE is parallel to KC;

$\therefore$ BGCK is a parallelogram, so that its diagonals must bisect each other;

$\therefore$ BD = DC,

so that the (*third*) line AD is a median too, etc.

Also GD = $\frac{1}{2}$GK = $\frac{1}{2}$AG, etc.,

and so the medians trisect each other.

The point of concurrency is the **centre of gravity (C.G.)** of the triangle. It is sometimes called the **centroid**. Usually the letter G is reserved for that position.

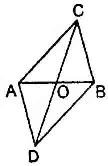
* The full proposition is given in the Appendix, Proposition C, page 227.

EXAMPLES 13.

(All Riders with the exception of the first.)

1. Sketch figures of *any* quadrilateral, parallelogram, rectangle, etc., being careful to make your sketches general (*i.e.* for a parallelogram do not draw a figure that might preferably be called a rhombus, or you might be tempted to generalize from a special case). Make a table like that on the following page and fill it up with Y. or N. (Yes or No), and for the first symmetry column give the number of axes of symmetry.

2. AB and CD are 2 straight lines which bisect each other at O. Prove that ACBD is a parallelogram.

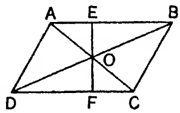


3. A quadrilateral ABCD has its opposite sides equal; prove that it must be a parallelogram.

[HINT. Join BD. Prove Δs ABD and CDB congruent.]

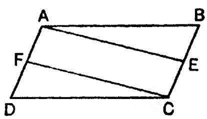
4. The diagonals of a parallelogram intersect at O. Any line is drawn through O to cut a pair of opposite sides in E and F. Prove OE = OF.

[HINT. Take triangles AOE, COF. Recollect that EOF is *any* line, so do not suppose it must be at right angles to AB; of course it is better to draw a "general" figure.]



5. ABCD is a parallelogram and E, F are the middle points of BC and AD respectively. Show that AE is equal to CF.

[HINT. Take triangles ABE, CDF.]

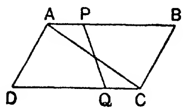


6. PQRS is a parallelogram; X is the mid-point of PQ; RX and SP are produced to meet at Y. Prove that PY and PS are equal.

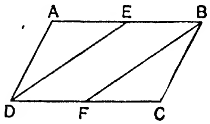
	Opposite sides parallel.	Opposite sides equal.	Opposite angles equal.	Diagonals equal in length.	Diagonals at right angles.	Diagonals bisect the area of the figure.	Diagonals bisect the angles through which they pass.	Symmetrical with regard to a line, in its plane.	Symmetrical with regard to a point.
Quadrilateral -									
Parallelogram -									
Rectangle -									
Rhombus -									
Square - -									
Trapezium -									
Isosceles Trapezium -									
Kite - -									

7. ABCD is a parallelogram and P and Q are points in AB and CD respectively such that AP = CQ. Prove that the straight line PQ passes through the point of intersection of the diagonals AC and BD.

[HINT. Only take one diagonal AC, and prove that PQ bisects that. Then say "similarly," etc.]



8. The sides AB, CD of a parallelogram ABCD are bisected at E, F, and ED, BF are joined. Prove that EBFD is a parallelogram.



9. Prove that the diagonals of a rhombus must bisect each other at right angles.

10. Prove that the bisectors of two opposite angles of a parallelogram are parallel to each other.

11. ABCD is a framework made of rods loosely jointed together; if AB = CD and AD = BC, show that when AB is kept fixed all positions of CD are parallel to one another.

12. Prove that the bisectors of the angles of a parallelogram must form a rectangle.

13. Prove that the diagonals of a kite must be at right angles.

14. If the diagonals of a parallelogram be equal, the parallelogram must be a rectangle.

15. D is the middle point of the side AB of a triangle ABC, and DE, DF are drawn parallel to the other two sides, cutting AC in E and BC in F. Prove that, if EF be joined, the four triangles so formed are equal in all respects.

16. ABC is a triangle. D is the fourth vertex of the parallelogram which has AB and AC as adjacent sides, and E is the fourth vertex of the parallelogram which has BA and BC as adjacent sides. Show that the three points D, C and E are in a straight line, and that C is the middle point of DE.

17. ABCD is an isosceles trapezium with AB parallel to DC, and AD = BC. Prove $\angle C = \angle D$.

[HINT. Draw perpendiculars from A and B on to DC.]

18. ABCD is a trapezium with AB parallel to DC. If $\angle C = \angle D$, prove $AD = BC$.

[This is the converse of the preceding. A direct proof should be obtained by producing DA and CB to meet, considering two isosceles triangles.]

19. Give the definition of a parallelogram.

Prove that if a 4-sided figure has all its sides equal it must be a parallelogram.

What is the name of a 4-sided figure with all its sides equal? What is its most important property?

20. ABCD is a quadrilateral in which $AB = CD$ and BC is parallel to AD. Must ABCD be a parallelogram? Give reasons, and illustrate your answer with figures.

21. ABCD is a parallelogram. The angle BAD is bisected by a straight line which cuts DC in X and BC produced in Y. Prove $BA = BY$ and $CX = CY$.

22. ABC is an isosceles triangle. D is any point in the base BC, and from D perpendiculars DH, DK are drawn on to AB and AC; prove that the sum of DH and DK is equal to the perpendicular from B on to AC.

* **23.** ABCD is a parallelogram. Points P, Q, R, S are taken on the sides AB, BC, CD and DA respectively, so that $AP = CR$ and $CQ = AS$. Prove that PQRS must be a parallelogram.

* **24.** Parallel to the base BC of a given triangle ABC draw a straight line, terminated by the sides of the triangle, and equal to a given straight line X.

[HINT. On BC mark off X and complete a parallelogram.]

* **25.** From a point A of a given straight line AB two equal straight lines AL, AM are drawn at right angles to AB and on opposite sides of it: and from B two equal straight lines BP, BQ are drawn at right angles to AB and on opposite sides of it. Prove that LP is equal to MQ.

[HINT. Through P and Q draw parallels to AB.]

* **26.** If AB and AC are respectively equal and parallel to DE and DF, prove that BC is equal and parallel to EF.

* **27.** State the names of parallelograms in which (*a*) the diagonals are equal, (*b*) the diagonals bisect the angles, (*c*) the adjacent sides are equal.

* **28.** A figure ABCD has the angles at B 120° and at A 60°, and is such that AD = AB + BC. Prove that (1) AD is parallel to BC, (2) AB = CD.

* **29.** PQRS and PQXY are two parallelograms on the same base PQ. (Of course they need not be of the same height.) Prove that SRXY must be a parallelogram.

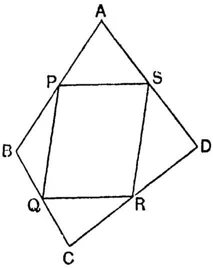
* **30.** ABCD is a parallelogram, and its diagonal CA is produced to E so that CA = AE.

If the parallelogram BAEF be completed, prove that FA and AD are in one and the same straight line.

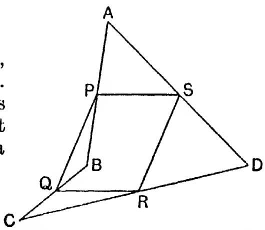
* **31.** The diagonal AC of a parallelogram ABCD is produced to E so that CE = CA; through E, EF is drawn parallel to CB to meet DC produced in F. Prove that ABFC is a parallelogram.

* **32.** ABCD is *any* quadrilateral, and P, Q, R, S are the mid-points of the sides. Prove PQRS must be a parallelogram.

[HINT. Join BD. Why must PS be parallel to BD? etc.]



* **33.** ABCD is *any* quadrilateral, in which ∠ABC is re-entrant. P, Q, R, S are the mid-points of the sides. As in the last rider, prove PQRS must be a parallelogram.



* **34.** Show that the straight lines which join the middle points of opposite sides of a quadrilateral bisect one another.

[HINT. Remember the joins of the mid-points of the sides, in order, make a parallelogram as in the two preceding questions.]

** 35. Prove that the lines joining the middle points of the sides of a rhombus taken in order form a rectangle.

** 36. BAC is a right-angled triangle (A the right angle). On BC a square BCED is described, and on AC a square ACGF is described (both squares are external to the triangle). EN is drawn perpendicular to AC produced. Prove that CGNE is a parallelogram.

** 37. On the sides AB, BC of a parallelogram ABCD equilateral triangles ABP, BCQ are described externally. Show that PQD is an equilateral triangle.

** 38. ABCD, AB′C′D′ are two parallelograms; prove that a triangle can be constructed so as to have its sides equal and parallel to BB′, CC′, DD′.

** 39. AD is the median of a triangle ABC. The triangle is drawn on paper and is folded across AD so that C falls at C′ in the same plane as ABC. The paper is now folded again so that B falls on C′. Prove that the second crease must pass through D and be perpendicular to AD.

CHAPTER XIV.

AREAS.

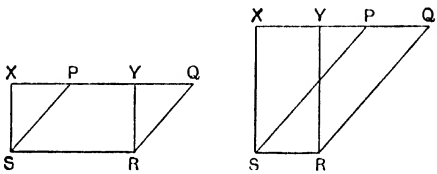
§ **1.** The method of calculating the areas of **rectangles** and **squares**, from their sides, is familiar.

§ **2.** The connexion between **parallelograms** and rectangles is next considered.

PROPOSITION 11.

General Enunciation. *The area of a parallelogram must be equal to that of a rectangle on the same base and of the same altitude.*

Particular Enunciation. PQRS is the parallelogram and RSXY the rectangle on the same base RS, and of the same altitude (that is between the same parallels). To prove that the parm. PQRS = rect. RSXY in area.



Proof. (*N.B.*—The same proof is required for the two possibilities indicated by the figures.)

In the two right-angled triangles SXP and RYQ,

$$\because \begin{cases} SX = RY \text{ (rect.)}, \\ SP = RQ \text{ (parm.)}, \\ \angle SXP \text{ and } \angle RYQ \text{ are right angles}, \end{cases}$$

$\therefore$ the triangles SXP and RYQ are congruent, and, in particular, their areas are equal.

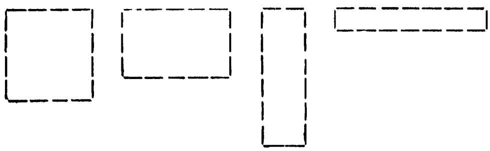
Take these areas, in succession, away from the whole figure XQRS, and the remainders, the parm. PQRS and the rect. RSXY, must be equal (in area). **Q.E.D.**

N.B.—If **boundaries** are equal, then **areas** need not be equal; comparing England and Wales with Scotland it is seen how wholly disproportionate are the figures given in the following approximate table:

	Area in sq miles	Coast line in miles.
England and Wales	58,000	2350
Scotland	30,000	2300

With nearly the same length of coast line the areas differ widely.

A rectangular pen for a few sheep is bounded by exactly 16 hurdles. The following figures show that though the *perimeters* are the *same*, the *areas* are very *different.*



We arrive at much the same conclusion if we consider the following rectangles, which have the *same area*, but *varying perimeters.*

36 yds. by 1 yd.,	area 36 sq. yds.,	perimeter	74 yds.
18 yds. by 2 yds.,	„	same	„	40 yds.
12 yds. by 3 yds.,	„	same	„	30 yds.
9 yds. by 4 yds.,	„	same	„	26 yds.
6 yds. by 6 yds.,	„	same	„	24 yds.

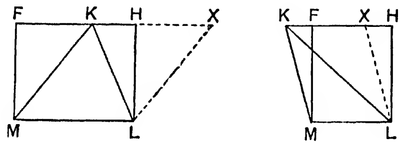
As a most important sequel to Proposition **11** it follows that **Parallelograms on the same or equal bases and of the same altitude are equal in area** (for each is equal to the same **rectangle).**

§ 3. The following proposition explains how the area of a **triangle** can be obtained from the area of a rectangle.

PROPOSITION 12.

General Enunciation. *The area of a triangle must be half that of the rectangle on the same base and of the same altitude.*

Particular Enunciation. KLM is the triangle and FHLM the rectangle on the same base LM, and of the same altitude. To prove that $\triangle$KLM $=\frac{1}{2}$ rect. FHLM in area.



Construction. Complete the parallelogram LMKX by drawing LX parallel to MK and producing KH, if necessary.

Proof.

N.B.—The proof applies to either figure. You must understand both. The second is very important.

The diagonal LK must bisect the parm. LMKX;

$\therefore$ $\triangle$KLM $=\frac{1}{2}$ parm. LMKX;

but the parm. LMKX = rect. FHLM in area, for they are on the same base LM, and of the same altitude FM;

$\therefore$ $\triangle$KLM $=\frac{1}{2}$ rect. FHLM in area. **Q.E.D.**

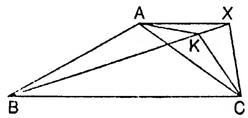
N.B.—Since any triangle is equal in area to half the rectangle on the same base and of the same altitude, it follows that **triangles on the same base (or equal bases) and of the same altitude are equal in area.** If the triangles are between the same parallels, the altitudes are automatically equal. (This is the most frequent case.)

PROPOSITION 13.

§ 4. **General Enunciation.** *Triangles, which are equal in area, and which are on the same base, must be of the same altitude.*

[The case where the triangles are on the same side of the base only is considered below. If the triangles are on opposite sides, turn one triangle over.]

Particular Enunciation. ABC and XBC are the triangles, of equal area, on the same base BC. To prove the altitude of $\triangle$ABC = altitude of $\triangle$XBC.



[Since the $\triangle$s ABC, XBC are on the same side of BC it will suffice to show them between the same parallels.]

Construction. If AX is not parallel to BC, through A draw AK parallel to BC, cutting BX (produced if necessary) at K. Join KC.

Proof. $\triangle$ABC = $\triangle$XBC in area (given),

and $\triangle$ABC = $\triangle$KBC in area (same base and same altitude);

$\therefore$ $\triangle$XBC = $\triangle$KBC in area,

that is, the whole is equal to the part,

and this is impossible,

so that AX must be parallel to BC,

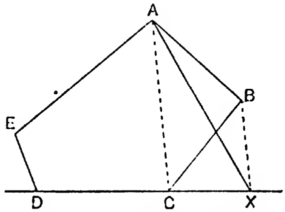
i.e. $\triangle$s ABC and XBC must be of the same altitude.

Q.E.D.

N.B.—If the triangles are on "equal" bases, move one and put it on the "same" base as the other.

§ 5. Reduction to a Triangle.

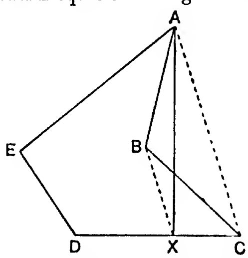
Figures bounded by straight lines can always be divided up into triangles, and the area of the figure can be found by totalling up the areas of all the triangles. The construction for making a figure equal in area to a given figure, but having one less side, is as follows. To fix our ideas, suppose that the original figure has 5 sides, and that it is required to construct a figure of only 4 sides, but having the same area.



Join AC, and through B draw BX, parallel to AC, to meet DC (produced if necessary) at X.

Then the figure AXDE is equal to the figure ABCDE in area, but has one less side.

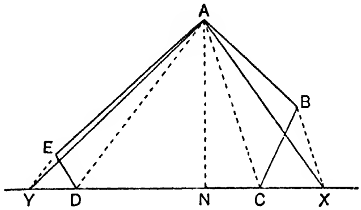
The proof consists in noticing $\triangle ACX = \triangle ACB$ in area (for they are on the same base AC and between the same parallels AC and BX); then, in adding to each of these the figure ACDE, getting the figure AXDE equal to the figure ABCDE. **Q.E.F.**



The case where the angle B is re-entrant is precisely similar, and letter for letter the same construction and proof hold (but there will be subtraction instead of addition).

By carrying on a similar construction, as indicated below, we should reduce the 5-sided figure ABCDE to a triangle AXY of equal area.

To find its area the perpendicular AN, from one of its angles to the opposite side, is needed too. Hence the area of ABCDE $= \frac{1}{2}$ AN $\times$ XY.



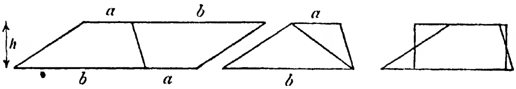
To find an area accurately by this method is difficult: it requires some thought as to how the reduction will come best on to the paper, and "good intersections" are absolutely necessary for good results. To find the area both by the sum of many triangles and also by reduction to a single triangle gives us two results, and if they confirm each other reasonably their average is fairly reliable.

The sum of many triangles can be done in more than one way; the reduction to a single triangle can also be done in more than one way. So we may have several results from which to take the average (at first of course rejecting any that are palpably wrong).

§ 6. For a **rhombus** the great thing to remember is that the diagonals must bisect one another at right angles, and the use of this fact often makes the finding of its area short.

The diagonals of a **kite** are at right angles too.

*§ 7. The area of a **trapezium** we can think out in several ways. a and b are the lengths of the parallel sides and h the distance between them.



(i) A second trapezium, like the first but upside down, makes a parallelogram whose area is $h(a+b)$, and hence the area of the trapezium $= \frac{1}{2}h(a+b)$.

(ii) A diagonal divides the trapezium into two triangles and the area is $\frac{1}{2}ha + \frac{1}{2}hb$ or $\frac{1}{2}h(a+b)$.

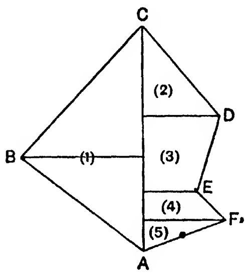
(iii) By bisecting the non-parallel sides we can make a rectangle of the same area. Its length is the average of a and b, *i.e.* $\frac{1}{2}(a+b)$; and hence its area is $\frac{1}{2}h(a+b)$.

*§ 8. The most common trapezium has one of its sides automatically perpendicular to the parallel sides. Its area is obtained without any construction at all. Parts of areas from **Field Book** notes are trapezia like this.

As an example to find the area of a field ABCDEF whose dimensions are given in the notes accompanying :

All dimensions in yards.

	C	
	190	
	145	70 D
B 90	105	
	90	60 E
	50	80 F
	A	



[The figure is purposely not drawn to scale.]

[Copy out the "notes" given overleaf.]

First draw a straight line AC to represent the 190 yds. (It need not be drawn to scale.) From A along it take 50 yds. (again not necessarily to scale, but at a sensible distance so as not to offend the eye), and, at right angles, to the right, take 80 yds. Now from A (the whole way and not from where we stopped before) 90 yards, and 60 yards to the right, to E, and so on. Joining up the points A, B, C, D, E, F will give us the area, which we consider as three triangles (1), (2), (5) and two trapezia (3) and (4).

The areas are

			Sq. yards.
$\triangle$ (1)	$\frac{1}{2} \times 190 \times 90$	=	8550
$\triangle$ (2)	$\frac{1}{2} \times 45 \times 70$	=	1575
Trapezium (3)	$\frac{1}{2}(70+60) \times 55$	=	3575
Trapezium (4)	$\frac{1}{2}(60+80) \times 40$	=	2800
$\triangle$ (5)	$\frac{1}{2} \times 50 \times 80$	=	2000
			18500

Area of ABCDEF = 18500 sq. yds.

Surveyors make entries in "Field Books" for small areas on this plan.

EXAMPLES 14 a (GEOMETRICAL DRAWING).

Rectangles and Parallelograms.

1. What are the length and breadth, in cm., of a rectangle 5″ by 4″? Determine its area in sq. cm. Calculate the number of sq. cm. in 1 sq. inch.

2. Construct a rectangle ABCD, where AB = CD = 3″, AD = BC = 3·5″. Join AC and draw a perpendicular BM on to AC and measure it. What is the area of ABCD ?

3. Draw a parallelogram with sides 3″ and 2″, and one of its angles 50°. Find its area.

4. Draw a parallelogram with sides 2·3 in. and 3·8 in. long, and one of its angles 143°. Mark on your figure the sizes of the angles of the parallelogram. Find its area.

5. Prick off the points P, T, Q (last page). Join PT and TQ (not PQ). Complete the parallelogram of which PT and TQ are two sides, using a set-square for the parallels. What is the area of the parallelogram in square centimetres ?

6. An equilateral parallelogram has each side 30 feet and one diagonal 22 feet. Find its area.

[HINT. Draw rough figure first on which to mark dimensions.]

7. The diagonals of a parallelogram are 3·2″ and 4·4″, while the angle between those diagonals is 50°. Draw that parallelogram correctly, explaining your process. Find its area too.

8. Make a parallelogram with diagonals 10 cm. and 6 cm. and with angle between them 50°. Find its area.

*9. Two parallel lines 3 inches apart are cut by two other parallel lines 2 inches apart at an angle of 45°. Find the area of the figure enclosed by the four lines.

*10. Draw a parallelogram of area 7 sq. in. on a base of 3·5 inches, having one diagonal 2·8 inches. Measure the other diagonal.

On the same base, draw also an isosceles triangle of the same area, and measure one of its sides.

[HINT. First calculate the height and draw a parallel to the base.]

*11. Draw a parallelogram whose sides are 6 cm. and 4·1 cm., and with each of one pair of angles two thirds of each of the other pair. Construct a triangle of the same area with two sides 6 cm. and 8·5 cm. respectively. What is the area ?

*12. Draw a parallelogram with sides 7 cm. and 10 cm. in length whose area is equal to that of a square on the shorter side. Measure the acute angle of the parallelogram.

*13. A parallelogram has a base 5 cm. and area 23 sq. cm.; what is its height ? Construct the parallelogram so that one of its angles is 50° ; on the same base construct a rhombus of equal area and measure its acute angle.

*14. On a base 3·5 inches long, draw a parallelogram of angle 50° and height 2 inches. On the same base construct a rhombus of equal area. Measure the acute angle of the rhombus.

*15. Construct a rhombus whose area is 8 sq. in. and whose sides are each 3·2″ long. Measure its smaller angle.

Triangles.

16. Draw a triangle with sides 3″, 2·8″ and 2·4″. Draw a perpendicular to the base from the angle opposite the base. What is the area of the triangle ?

17. Prick off the points A, F, C (last page). Find the area of the triangle AFC in square centimetres.

18. I wish to run a fence BC across the corner of a field so as to enclose a triangular piece of ground ABC. AB is 5·72 chains and AC = 8·24 chains, while the angle BAC = 63°. Find (by measurement) the length of the fencing required and the acreage of the ground enclosed.

19. ABC is a triangle having AC = 7·2″, BC = 9·6″. AX, BY are drawn perpendicular to BC, AC respectively. If AX = 2·4″, find the length of BY.

20. Draw an isosceles triangle with base 2″ and base angles 64°. On the same base make an isosceles triangle of double its area. What are its base angles ?

21. Draw a triangle with sides 6 cm. and 5 cm. and included angle 110°. Find its area.

22. Construct a triangle ABC, the area being 24 sq. cm., the side BC 8 cm. long, and the angle CBA 67°.

Also find, by measurement, the other sides and angles.

23. On a base QR 10 cm. long, construct a triangle PQR of area 42·5 sq. cm. and having PQ 9 cm. in length. Measure the angle PQR.

***24.** ABC is a triangle in which BC, BA have constant lengths 6 cm. and 5 cm. If BC is fixed and BA revolves about B, find by drawing the values of the area of the triangle ABC when B = 0°, 30°, 60°, 90°, 120°, 150°, 180°. With the aid of the results you obtain draw a graph to illustrate the change in area, and so find for what value or values of the angle B the area is 10 sq. cm.

Also find for what value of B the area is a maximum.

***25.** Construct an isosceles triangle whose base is 10 cm. and equal sides 13 cm.

Construct a parallelogram on the same base having a base angle of 60° and its area three times that of the triangle.

Measure the areas of both figures and write them down.

*** 26.** Prick off the corners of the figure PQRST (last page). Join up the points and also join PR and PS. Find the area of each triangle in square centimetres and so of the whole figure PQRST.

*** 27.** A triangle has its base 2·8 inches long, and is of area 3·5 square inches. Calculate its height. One of its sides is 4·7 inches long. Construct the triangle, and measure its greatest angle.

Polygons : Reduction to Triangle.

*** 28.** Prick off the corners of the figure PQRST (last page). Reduce the figure to a triangle of equal area, and so obtain the area of the original in square centimetres. [How do your answers for this and for question 26 agree ?]

*** 29.** Draw a quadrilateral ABCD with $DA = 1''$, $AB = 2{\cdot}2''$, $BC = 1{\cdot}5''$, $\angle A = 100°$, $\angle B = 64°$. Reduce the quadrilateral to an equivalent triangle, and find its area.

*** 30.** Construct a quadrilateral ABCD in which $AD = 2''$, $AB = 3{\cdot}2''$, $\angle A = 110°$, $\angle B = 33°$, $\angle D = 74°$. Reduce it to a triangle and find its area.

*** 31.** Draw a plan of a four-sided field ABCD, in which $AB = 100$ yards, $\angle ABC = 75°$, $BC = 120$ yards, $\angle BCD = 115°$ and $CD = 115$ yards. (Make 1 cm. represent 10 yards.) Also find the area of the field.

*** 32.** ABCD is a quadrilateral in which the diagonal $BD = 3{\cdot}7$ in., $\angle ADB = 65°$, $\angle DBA = 40°$, $\angle CBD = 30°$, $\angle BDC = 80°$. Reduce it to a triangle, and hence find its area.

*** 33.** A field ABCD has $AB = 348$ yds., $BC = 195$ yds., $CD = 187$ yds., $\angle B = 93°$, $\angle C = 107°$.

Draw a plan of this; and by measurements from your plan find the length of a path from C to A, and the area of the field in acres.

*** 34.** Draw a plan of a field ACDE in which $AE = 260$ yds., $AC = 240$ yds., $CD = 140$ yds., $CE = 210$ yds., $DE = 180$ yds.

Find the area of the field, reducing your figure to a triangle.

*** 35.** A man wishing to find the area of a four-sided field began at one corner to walk round the field with a compass in his hand. He first had to walk 250 yards due N., then 150 yards N.N.E. From that point he went S.E. for 325 yards, and then went home and worked out the result. What ought he to have found for the length of the fourth side, and what was the correct area ?

∗ **36.** Draw an isosceles triangle having its base 2″ and vertical angle 30°. Construct a triangle of equal area having its base 2″ and one of the base angles 50°.

Measure the longest side of the second triangle.

∗ **37.** Construct a $\triangle ABC$ having $\angle A = 80°$, $\angle B = 65°$, $a = 10$ cm.

Take a point D in BC such that BD = 7 cm.

On BD construct a triangle equivalent to the $\triangle ABC$; having BA produced as one side.

Find a point E in AB such that DE will bisect the $\triangle ABC$. Measure BE.

∗ **38.** Prick off the points A, B, C, D, E, F (last page). Reduce the figure ABCDEF to a triangle of equal area and find its area in square centimetres.

Diagonals at right angles.

∗ **39.** Draw a rhombus with sides 6 cm. and with one angle 55°. Find its area.

∗ **40.** Draw two straight lines AC and BD at right angles. Join up to get the quadrilateral ABCD. Find its area (i) by an easy calculation (for its diagonals are at right angles), (ii) by reduction of the quadrilateral to a triangle. Do the two results agree reasonably ?

Trapezia.

∗ ∗ **41.** Draw a trapezium with parallel sides 6 cm. and 11 cm. and the other two sides 5·2 cm. and 4·8 cm.; find its area. [Hint. Subtraction of the lengths of the parallel sides will give you the base of a triangle. Draw the triangle first and then the parallelogram.]

∗ ∗ **42.** Draw an isosceles trapezium with parallel sides 4·7″ and 3·1″, and with non-parallel sides each 2·5″. Find its area.

∗ ∗ **43.** A piece of ground is in the form of a trapezium. The lengths of the parallel sides are 20 yards and 34 yards, and the lengths of the other two sides 15 and 13 yards. Find by drawing to scale its longest diagonal.

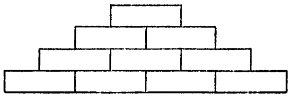
EXAMPLES 14 b (CALCULATIONS).

Many suitable questions can also be found in Arithmetic Books.

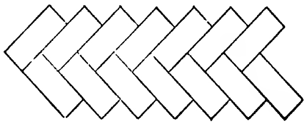
Rectangles.

1. A towel is 4′ 6″ by 2′ 6″. Sketch it and calculate its area.

2. Each of the bricks is 9″ long by $2\frac{1}{2}$″ high. Find the total area.



3. Bricks, 9″ by $3\frac{1}{2}$″, are arranged in the herring-bone pattern. What is the total area shown ?



4. The letter T is composed of two rectangles, the horizontal $0{\cdot}9'' \times 0{\cdot}15''$, and the vertical $1{\cdot}1'' \times 0{\cdot}15''$. What is the total area of the letter ?

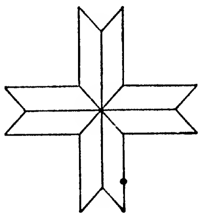
5. An L-shaped lawn is two rectangles, one 20 by 10 yards and the other 15 by 5 yards. What is the combined area ? There are two ways in which they may be situated to form the L. Sketch those two ways, and give the perimeter in each case.

6. Two rectangles, 20 yards by 4 yards and 14 yards by 3 yards, are put together like the letter L. Sketch the two possible ways and give the perimeter in each case.

Also, what is the area ?

Parallelograms.

7. Eight equal parallelograms form the cross shown. The length of the central lines are 20″ (*i.e.* 10+10) and the width of the arms of the cross are 6 inches. What is the area of the cross ?



8. Calculate the area of a parallelogram whose sides are 7″ and 4″ respectively, the longer sides being 3″ apart.

9. The sides of a parallelogram are 18 feet and 8 feet. The perpendicular distance between the two longer sides is 5 feet. Calculate the distance between the two shorter sides.

10. Two adjacent sides of a parallelogram are 8·4 inches and 12·6 inches. The distance between the two sides 8·4 inches long is 5·2 inches. Calculate the distance between the two sides 12·6 inches long.

11. The block capital **X** is made up of two parallelograms crossing each other. The area of the piece common to both is 1 sq. in. The total height of the letter is 5 in. The width of each arm, measured along the base line (and therefore perpendicular to the height) is 1·4 in. Find the area of the letter.

Triangles.

12. A square, whose sides are 10″, is divided into 4 equal triangles by its diagonals. What is the area of each of the triangles ?

13. ABCD is a rectangle 6″ by 4″. X is some point in the side AB. What is the area of the triangle CXD ?

14. A parallelogram with long sides 12 ft., and 5 ft. apart (perpendicularly), is divided into 4 triangles by its diagonals. What is the area of each of the triangles ?

Diagonals at right angles.

15. The diagonals of the "Ace of Diamonds" (a rhombus) are 1·6 cm. and 1·3 cm. What is its area ?

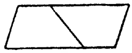
16. The "Nine of Diamonds" is a white card 8·9 cm. by 6·4 cm. On it there are 9 red pips of the same size as in the preceding question. What is the white area ?

17. The diagonals of a quadrilateral are at right angles, and are 4 in. and 3 in. long. What is the area of the quadrilateral ?

Trapezia.

*** 18. The block capital letter V is made up of two equal trapezia, each 3 mm. wide. The slant lengths are outside 2·6 cm. and inside 1·8 cm. What is the area of the letter ?**

*** 19.** Two equal trapezia together form a parallelogram whose total length is 14″. The parallel sides are 6″ apart. What is the area of one trapezium ?

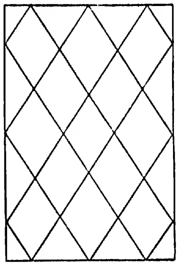


*** 20.** Construct a trapezium ABCD in which AB is parallel to DC, AB = 1·8 cm., AD = 4·6 cm., CD = 5·4 cm., angle D = 47°. Measure the distance of AB from DC, and use your result to calculate the area of the trapezium.

Miscellaneous.

21. Find the area of the black in the Greek key pattern shown. (The width of the strips, black and white, is everywhere to be reckoned ½ inch.)

22. A window is glazed with diamond-shaped panes of glass. The window (shown) is 45 cm. by 30 cm. How many full panes are there ? How many half panes ? How many quarter panes ? What is the area of a full pane ?



23. Equal squares are cut from the four corners of a rectangular sheet of cardboard, and the sides and ends are turned up so as to form an open rectangular box. Find the cubical content of boxes formed in this way from sheets measuring each 30 inches by 14 inches, when the squares cut out are of 1 inch side, then of 2 inches side, and so on up to 5 inches side.

Plot a graph showing the relation between the cubical content of the box and the side of the square cut out, and determine, approximately, the side of square which corresponds to the largest box.

24. A kite, of the usual shape, has diagonals 3′ 2″ and 2′ 7″. What is its area ?

25. The block capital letter N is made up of two rectangles 5 in. by 0·4 in. and an oblique parallelogram. This parallelogram is of the same width as the rectangles and (slopingly) the parallelogram is 6·3 in. long. What is the area of the letter?

26. The block capital letter M is made up of two rectangles (3 cm. by 2 mm.) and two parallelograms (each 2 mm. wide and of oblique length 3·5 cm.). What is the area of the letter?

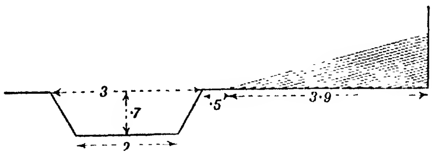
* 27. The arch (figure on page 140) is built with equal stones whose longitudinal cross sections as shown are trapezia. The lengths of the sides of the stones are outside 15 in., inside 12 in., and the outside and inside are 6 in. apart. What is the area of this longitudinal cross section?

* 28. One diagonal of a four-sided field is 426 yards long, and the perpendiculars from the other two corners upon this diagonal are 170 yards and 266 yards respectively. Calculate the area of the field in acres; also calculate what area it covers on a plan drawn to a scale of 1 inch to 100 yards

* 29. Draw a rhombus whose diagonals are 7·6 and 4·2 cm. Calculate its area, and construct, on one of its sides, an isosceles triangle of equal area.

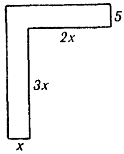
* 30. Construct a quadrilateral ABCD in which AC = 3·5″, AB = 2″ and BC = 2·5″, also ADC is a triangle such that BD is perpendicular to AC and 4″ long. Calculate the area of ABCD.

* 31. The figure is a *rough* sketch of the section of a trench, out of which earth has been dug and piled up against a vertical wall, at an uniform slope, the earth occupying 30 % more space



in its new position. Using mensuration formulae determine the height of the earth in contact with the wall; and, on squared paper, draw the figure to scale.

[All the dimensions in the figure are metres.]

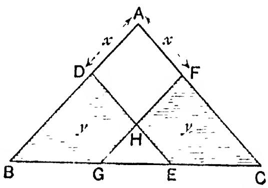


32. In the accompanying figure (which is not drawn to scale) all the dimensions are in inches, and the total area is 53 sq. ft. Find x.

33. A trapezium has one of its parallel sides 14 cm. longer than the other, while the perpendicular distance between them is 3 cm. less than the shorter parallel side. The area of the trapezium is 119 cm². Calculate the length of the shorter parallel side.

34. The cross-section of a pillar is a regular hexagon of side 10 inches, and its height is 32 feet; find the area of its surface, excluding the top and bottom, and its volume. [You may assume that, if the edge of an equilateral triangle is a, its area is $\frac{a^2\sqrt{3}}{4}$].

35. ABC (in the figure) represents a flower-bed in the form of an isosceles right-angled triangle whose sides AB, AC are each 12 feet long. Parallels DE, FG are drawn at equal distances x feet from AC, AB respectively, to divide ABC into four areas, those shaded being each y square feet.



Express the lengths BD, GF, GH in terms of x, and then express the four separate areas in terms of x alone. Draw a graph showing y as a function of x, and carefully mark the scales.

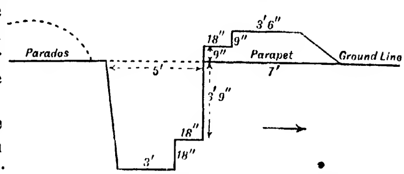
Hence, or otherwise, find how x must be chosen to make y as large as possible.

36. The figure represents a section of a fire trench.

(1) Calculate the area of the section excavated.

(2) Assume the loose earth will occupy 20 per cent. more space than before.

(3) Calculate the area of the section of the parapet.



(4) What earth is available for making the parados? Give the area of its section.

* **37.** A railway embankment is 36 feet wide at the top, x feet in vertical height, with sloping banks such that 4 feet vertically correspond to 5 feet along the slope.

(*a*) Draw a plan of its cross section when the bank is 16 feet high.

(*b*) Calculate the volume of earth in 100 feet of the embankment when it is x feet high.

(*c*) Draw a graph to show the change in volume as x increases from 0 to 20.

(*d*) The earth is obtained by digging a ditch 20 feet deep, with perpendicular sides, alongside the line, of width y feet varying to meet the demands of the embankment: what width will the ditch be when the embankment is 8 feet high?

* **38.** ABCDE is a five-sided field and perpendiculars from B, D and E meet AC at K, L and M respectively.

Find the area (in acres) of the field from the following data:

AC = 210 yards, EM = BK = 52 yards, DL = 40 yards,

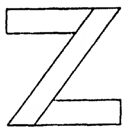
AM = ML = LC.

* **39.** The letter Z is made up of one parallelogram and two trapezia as is indicated.

The following dimensions are given:

Total height 1 in.

Total width 1·1 in. (0·3″ + 0·8″).



Length of shorter parallel side of the trapezia 0·7″.

Width of trapezium (perpendicular to parallels) 0·2″.

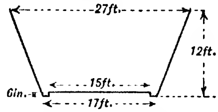
Find the area of the letter.

* **40.** The section of a trench is a trapezium, and the width at the top is 5 feet and at the bottom is 3 feet, while the depth is 6 feet. If one man is working, and has both a pick and shovel available, then the average number of cubic feet he can excavate in 1 hour is given by the accompanying table.

On the average, what length of this trench would one man dig if working for 5 hours?

1st hour.	2nd hour.	3rd hour.	4th to 8th hour.
30	25	15	10

* **41.** The accompanying sketch (which is not drawn to scale represents a cross section of a railway cutting 12 feet high. The part 15 feet broad (to carry the rails) is 6 inches high, thus leaving a water channel on each side for drainage. Calculate the area of the cross section, and check your result by a drawing to scale. If the cutting is 100 yards long, and a cubic foot of earth weighs about 120 lb., find the weight of the earth that has been removed, to the nearest 10 tons.



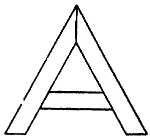
* **42.** The letter **A** is made up of three trapezia as is indicated. The following are its dimensions.

The slant sides are :

Externally 4·1 cm.
Internally 3·9 cm.

The bar is :

Upper 1·5 cm.
Lower 1·9 cm.



The width throughout is 0·7 cm.
Calculate the area of the letter.

* **43.** The mainsail of a yacht (about 15 tons size) is quadrilateral. One diagonal is 37·25 ft. and the perpendiculars to that diagonal from the other two corners are 18·25 ft. and 22·9 ft. Calculate the area of the sail.

* **44.** The mainsail of a yacht (about 30 tons size) is quadrilateral. One diagonal is 53·8 ft. and the perpendiculars to that diagonal from the other two corners are 23·2 ft. and 29·1 ft. Calculate the area of the sail.

* **45.** ABC is a triangle with a right angle at B, and BD is drawn at right angles to AC, meeting AC at D. If BC is a inches, CA is b inches, and AB is c inches long, prove by writing down two different expressions for the area of the triangle ABC that the length of BD is $\frac{ac}{b}$ inches. Test your result by an accurate figure, taking the values $a=3$, $b=5$ and $c=4$.

* **46.** Draw, *roughly*, a plan of a field whose dimensions are shown in the accompanying field notes. On each line mark its dimensions and calculate the acreage of the field.

All dimensions in chains.

	C	
	11·2	
	10·4	3·2 D
	9·6	2·8 E
B 4·2	7·2	
	A	

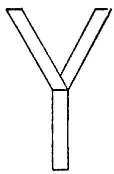
* **47.** Draw a plan of a field ABCDEF from dimensions given in the accompanying "Field Book." All the dimensions are in chains. Calculate the acreage of the field.

	C	
	10·4	
	8·2	6·0 D
B 4·8	7·4	
	6·2	5·4 E
	3·2	7·0 F
	A	

* **48.** Draw a plan of the field PQRSTV from the accompanying dimensions which are given in metres. The area in hectares is required. [10,000 sq. m. = 1 Hectare.]

	S	
	410	
	320	170 T
R 190	240	
Q 98	78	
	38	80 V
	P	

* * **49.** The letter Y is made up of a rectangle, parallelogram and trapezium. The total height is 3″. The height of the rectangle is $1\frac{1}{2}$″ and its width $\frac{1}{4}$″. The parallel sides of the trapezium are $1\frac{3}{4}$″ and $1\frac{1}{2}$″. The perpendicular width of the trapezium is $\frac{1}{5}$″. The dimensions of the parallelogram are like those of the trapezium. Calculate the area of the letter.



* * **50.** Draw two straight lines NOS and WOE at right angles to show the cardinal points of the compass. Mark the positions of the five points P, Q, R, X, Y, whose Northings, Southings, Eastings and Westings are given. Calculate the exact area of the polygon PQRXY in sq. units, by dividing it up into right-angled triangles and rectangles.

	N.	S.	E.	W.
P	12	—	2	—
Q	6	—	8	—
R	—	6	6	—
X	—	12	—	8
Y	4	—	—	12

51. Draw two straight lines NOS and WOE at right angles to show the cardinal points of the compass. Mark the positions of the six points A, B, C, D, F, G, whose Northings, Southings, Eastings and Westings are given in miles. Calculate the exact area of the polygon ABCDFG in sq. miles by dividing it into right-angled triangles and rectangles.

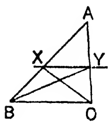
	N.	S.	E.	W.
A	5	—	2	—
B	1	—	7	—
C	—	2	3	—
D	—	4	2	—
F	—	2	—	3
G	3	—	—	2

52. What is the area of a triangle whose height is x'', and whose height and base together make up $10''$? Use this result to find (correct to one hundredth of an inch) the height of such a triangle if its area is 10 sq. in.

Draw figures to scale to illustrate your answers.

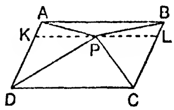
EXAMPLES 14 c (RIDERS).

1. A straight line XY is drawn parallel to the base BC of any triangle ABC cutting AB at X and AC at Y. Prove (i) that the triangles XBC and YBC are equal in area, and (ii) that the triangles ABY and ACX are equal in area.



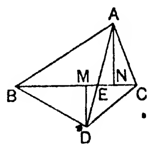
2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

[HINT. First draw KPL parallel to AB and DC; how big is ΔDPC compared to parallelogram KLCD? etc.]



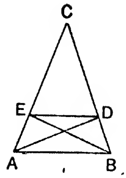
3. ABC, DBC are two triangles, equal in area, but not in other respects, on opposite sides of the same base BC. If AD cut BC at E, prove that AE=ED. (Of course the figure is not to scale.)

[HINT. Join AD and draw the two perpendiculars DM and AN.]



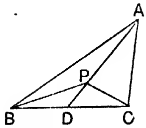
4. ABC is an isosceles triangle, with AC = CB, and D, E are the feet of the perpendiculars from A and B on BC, AC respectively. Prove that DE is parallel to AB.

[HINT. First prove that Δs ADB, AEB are congruent.]



5. ABC is any triangle. D is the middle point of BC. On AD any point P is taken. Prove that the triangle APB is equal to the triangle APC in area.

[HINT. ΔAPB = ΔABD − ΔBPD, etc.]

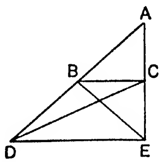


6. ABCD is a square and E a point on the diagonal BD such that BE = $\frac{1}{4}$BD. Prove that the area of the triangle CED is three times the area of the triangle AEB.

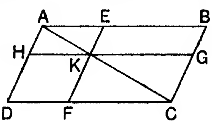
7. Show how to construct a parallelogram, given the lengths of two adjacent sides and the perpendicular distance between a pair of opposite sides.

*8. ABCD is a parallelogram. AE is drawn parallel to BD to meet CD produced in E; and CF is drawn parallel to BD to meet AD produced in F. Prove that the triangles ADE and CDF are equal in area.

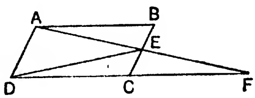
*9. ABC is any triangle; a straight line is drawn parallel to BC to cut AB and AC produced in D and E. Prove that the triangles ABE, ACD are equal in area.



*10. ABCD is a parallelogram, and through K, any point on the diagonal AC, a line EKF is drawn parallel to AD to meet AB in E and CD in F, and HKG parallel to AB to meet BC in G and AD in H. Prove fig. EG = fig. HF in area.



*11. ABCD is a parallelogram, E is the middle point of BC, and AE and DC produced meet in F; prove that the triangle DEF is half of the parallelogram ABCD.



*12. PQRS is a rhombus whose diagonals are PR and QS Show that a rectangle on base PR and of altitude QS would be twice as large as the rhombus.

*13. PQR is a triangle; it is required to bisect it by a straight line through X, a given point on QR. Justify the following construction for such a line.

Join PX; through D, the mid-point of QR, draw DY parallel to PX and cutting PR or PQ at Y and join XY: then XY is the required line.

*14. Construct a rhombus equal to a given triangle.

*15. PQRS and PLMN are two parallelograms, on the same side of a given straight line, which are equal in area. Besides having the common point P they are situated so that P, L and Q are in a straight line and P, S and N. Prove LS and RM must be parallel.

[HINT. Join RL and SM. Consider the areas of △s SRL and LSM.]

*16. E is any point on the diagonal AC of a parallelogram ABCD. Prove that the triangles ABE, ADE are equal in area.

*17. ABCD is a parallelogram. E and F are the mid-points of AB and CD. ED and FB cut the diagonal AC at G and H respectively.

Prove (1) EBFD is a parallelogram.

(2) $\triangle HFC = \triangle HFD = \triangle HFG$ in area.

(3) AC is trisected at G and H.

*18. ABCD is a parallelogram; through A a straight line AQP is drawn cutting BC in Q and DC produced in P. Prove that the triangles BQP and DQC are equal in area.

*19. The side BC of the parallelogram ABCD is produced to E so that CE=BC. If AE cuts CD at F, prove that the triangle ABF is double the triangle CFE in area.

*20. A triangle ABC is given: find a point P such that each of the triangles PAB and PAC may be equal in area to ABC.

*21. ABCD is a parallelogram, APQR any straight line through A cutting the diagonal BD in P and the sides CD, CB (produced if necessary) in Q and R. Join DR, BQ.

Prove that the area of the triangle DPR = area of the triangle APB, and the area of the triangle BPQ = area of the triangle APD.

*22. Any point D is taken in the side AC of a triangle ABC. Through B a straight line BO is drawn parallel to AC and

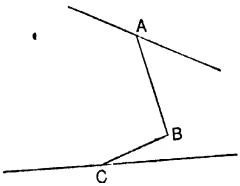
through D a straight line DXO is drawn parallel to BC to meet AB in X and BO in O. Show that the triangles CBX and AOB are equal in area.

* 23. ABC is a triangle with D, E, F the middle points of its sides. Prove that the triangle DEF is $\frac{1}{4}$ the triangle ABC.

* 24. Any point O is taken within a parallelogram ABCD; show that the triangle OBD is equal to the difference of the triangles OAB, OBC.

* 25. If p, q, r are the three perpendiculars from the angles of a triangle ABC on the opposite sides, prove that $p^2/qr = bc/a^2$.

* 26. Two men own fields separated by a hedge ABC, as in the diagram (which is not drawn to scale). They agree to replace it by a straight hedge starting at A. Determine geometrically how this may be done so that the areas of the fields are unaltered. State your construction carefully, and draw an accurate figure, showing the old and new hedge.



* 27. ABCD is a quadrilateral having AB parallel to CD; prove that the line joining the middle points of AB and CD divides the quadrilateral into two parts of equal area.

* 28. ABC is a triangle and R, Q the middle points of the sides AB, AC; show that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR in area.

* 29. Show how to bisect a quadrilateral by a straight line drawn through an angular point.

[HINT. Reduce quadrilateral to appropriate triangle first.]

* 30. ABCD is a parallelogram, in which AB is produced to O, so that BO = AB. The lines AD, OC, when produced, meet at P. Prove that triangles PCD, ACB are equal in area.

* 31. AB, CD are the parallel sides of a trapezium ABCD, and AD is bisected at E; prove that the triangle EBC is half the trapezium.

[HINT. Through E draw a parallel to CB and make a parallelogram.]

* 32. ABCD is a square. E and F are points in AB, BC respectively. AF and DE meet in G, and the figure EGFB is equal to the triangle AGD. Prove that BF = AE.

*33. ABCD is a parallelogram. The sides AB, AD are produced, and the exterior angles are bisected by lines which cut a line drawn through C parallel to BD in E, F.

Prove that BDFE is a parallelogram, and that its area equals that of ABCD.

*34. ABC, PQC are two straight lines such that the triangle AQC is equal to the triangle PBC. Show that BQ is parallel to AP, and that if X is any point in AP, the sum of the triangles XBC and XQC is equal to the triangle AQC.

**35. ABCD is a parallelogram, and a point O is taken outside the parallelogram, but between the parallel lines AB and DC when these are produced. Show that the difference between the areas of the triangles OAD and OBC is half the area of the parallelogram.

**36. ABC is a triangle and AB, AC are bisected in D and E respectively. BE and CD intersect in G. Prove that the triangles AGB and AGC are equal in area.

**37. The perimeter of an isosceles triangle is greater than that of the rectangle of the same altitude and of equal area.

**38. The base BC of a triangle ABC is divided into three equal parts at P and Q, and X is any point in PQ. Prove that ABP, APQ and AQC are all equal in area. Parallels to AX through P and Q meet the sides of the triangle at H and K. Prove that XH and XK also divide the area of the triangle into three equal parts.

**39. If P is a point within the triangle ABC such that the sum of the areas PAB, PAC is constant, show that the locus of P is a straight line.

**40. ABC is a triangle and D the middle point of AB; DE is drawn parallel to BC cutting AC at E.

What fraction of the area of the triangle ABC is the area of the triangle ADE? Give reasons for your answer.

If ABC were the cross section of a triangular wedge, and the top was cut off by a plane through DE and parallel to the base, what fraction of the wedge would remain?

**41. AB and ECD are two parallel lines; BF, DF are drawn parallel to AD, AE respectively; prove that the triangles ABC, DEF are equal to one another in area.

**42. The medians of a triangle ABC intersect at G; prove that the triangles GAB, GAC, GBC are equal in area.

CHAPTER XV.

PYTHAGORAS' THEOREM.

§ **1.** The discovery of the theorem which follows has always been ascribed to **Pythagoras** (about 500 B.C.) and is called after him. The proof is due to **Euclid** (about 300 B.C.) Many other proofs have been given, and some are indicated later. The truth of the proposition is of fundamental importance.

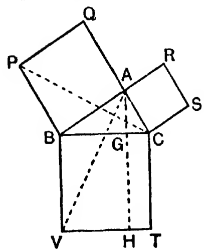
PROPOSITION 14.

§ **2. General Enunciation.** *The square described on the hypotenuse of a right-angled triangle must be equal to the sum of the squares on the other two sides.*

Particular Enunciation. ABC is the rt.-angled △ (BAC the rt. ∠).

The squares on the 3 sides are ABPQ, CARS and BCTV.

To prove $BC^2 = BA^2 + AC^2$.



Construction. Join PC and AV.

Draw AGH parallel to BV, meeting BC in G and VT in H.

Proof. (i) QAC is a straight line, for the adjacent angles QAB (rt. $\angle$ of a square) and BAC (given a rt. $\angle$) are together equal to two right angles.

(ii) In the $\triangle$s BAV, BPC,

$$\because \begin{cases} BA = BP \text{ (sides of a square)}, \\ BV = BC \text{ (sides of a square)}, \\ \angle ABV = \angle PBC \text{ (rt. } \angle + \angle ABC), \end{cases}$$

$\therefore$ the $\triangle$s BAV, BPC are congruent, and, in particular, equal in area.

(iii) $\triangle BAV = \frac{1}{2}$ rect. BVHG (same base BV, same altitude VH).

And $\triangle BPC = \frac{1}{2}$ square PBAQ (same base PB, same altitude PQ).

(iv) Now we have shown $\triangle BAV = \triangle BPC$;

$\therefore$ rect. BVHG = square PBAQ = BA^2.

Similarly rect. GHTC = square CARS = AC^2;

$\therefore$ by addition $BC^2 = BA^2 + AC^2$. **Q.E.D**

[The object in proving QAC a straight line, in (i), is to be able to say that the *whole* of QAC is parallel to PB, and consequently to make the second line of (iii) lawful.]

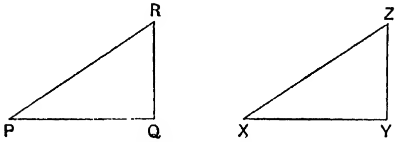
§ **3.** The converse of Pythagoras' Theorem is true. The following proof involves the principle of *Reductio ad absurdum.*

PROPOSITION 15.

General Enunciation. *If the square on one side of a triangle is equal to the sum of the squares on the other two sides the angle between these two sides must be a right angle.*

Particular Enunciation. PQR is the triangle, in which $PR^2 = PQ^2 + QR^2$. To prove that $\angle Q$ must be a right angle.

Construction. Make XY equal to PQ and YZ equal to QR, with $\angle Y$ a right angle.



[You must remember that $\angle Y$ is definitely, by construction, a right angle, but nothing is known about $\angle Q$.]

Proof. $XZ^2 = XY^2 + YZ^2$ (Pyth.);

but $XY = PQ$ (constn.)

and $YZ = QR$ (constn.);

$\therefore XZ^2 = PQ^2 + QR^2$;

but $PR^2 = PQ^2 + QR^2$ (given);

$\therefore XZ^2 = PR^2$,

or $XZ = PR$.

Now, in the triangles XYZ, PQR,

$$\because \begin{cases} XY = PQ \text{ (constn.)}, \\ YZ = QR \text{ (constn.)}, \\ XZ = PR \text{ (proved)}, \end{cases}$$

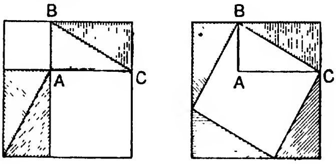
$\therefore$ the triangles XYZ and PQR are congruent,

and, in particular, $\angle Y = \angle Q$.

Now $\angle Y =$ rt. $\angle$. (constn.),

hence $\angle Q$ must be a right angle. **Q.E.D.**

§ 4. A different proof of Proposition 14 may be obtained as is indicated in the accompanying diagram.



ABC is the rt. ∠d △, of which ∠A is a rt. ∠.

In the left-hand figure (white) squares are described on BA and AC, and the whole big square completed.

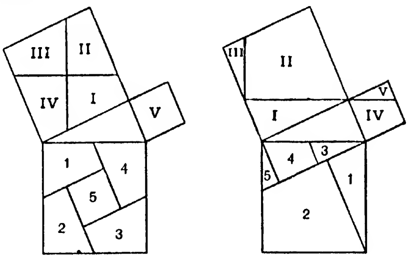
In the right-hand figure the (white) square on BC is described, and the whole big square is completed.

From each take the 4 equal △s (shaded alike);

$$\therefore\ BA^2 + AC^2 = BC^2.$$

[Above is a mere statement that various things, certainly looking suspiciously equal, are equal; but a mere statement is not a proof. A proper proof, on these lines, affords a good exercise.]

* § **5.** Many proofs of Pythagoras' Theorem have been given, among others **dissection proofs**. Two (of several) are indicated below.



For the former (by **Perigal**) find the mid-point of the bigger square, *i.e.* where its diagonals would cut, and through that point draw a parallel and perpendicular to the hypotenuse

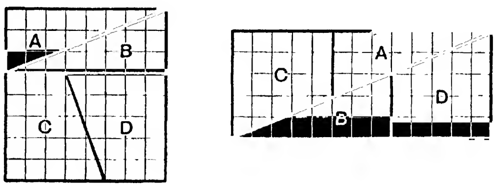
of the right-angled triangle. Cut out the five pieces with Roman numbers and fit them on to the square on the hypotenuse like a jig-saw puzzle.

For the latter the parallel and perpendicular are sufficiently indicated on the figure. Cut out the five pieces with Roman numbers. You should have no trouble in seeing how the lines on the square on the hypotenuse are obtained, except the line between 3 and 4, whose position may be got by measurement.

Of course it is quite possible to put letters at the corners of the various figures and to prove the proposition thus, but the proofs are more laborious than the time-honoured proof given.

Recollect that it is not a "proof" to say that they fit without substantiating your statement with reasons. Statements may or may not be true; they are not reasons.

* § 6. The following may show the **fallacy** in trusting too much to appearances.



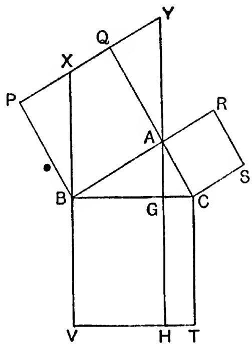
On the left is a square 8 by 8, *i.e.* 64 small squares, like a chessboard. On the right is a rectangle 13 by 5, *i.e.* 65 small squares. The 64 square is cut into four pieces A, B, C, D as is indicated, and those pieces are laid down on to the 65 rectangle (the C and D pieces will have to be turned through a right angle).

The "fit" seems all right; $\therefore$ $64 = 65$, and this is clearly wrong.

Where is the flaw?

The moral is, look with suspicion on dissection "proofs" without formal reasoning.

§7. A proof, perhaps due to the mathematician **Legendre (about 1800), may be obtained as is indicated in skeleton thus:



(i) Δs PBX, ABC are congruent.

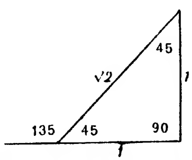
(ii) parm. XBAY = square PQAB
and rect. BGHV = parm. XBAY.

(iii) $\therefore$ rect. BGHV = square PQAB.

Similarly, etc.

Writing out the proof, with full reasoning, affords a good exercise.

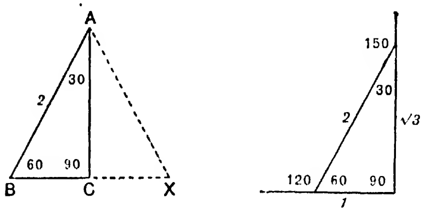
*§8. **An isosceles right-angled triangle** has sides and angles as indicated. [Pythagoras' Theorem is needed in the proof.]



The angles are **45°, 45° and 90°.** If one of the equal sides is produced we get the angle **135°.**

*§ 9. **If the angles of a triangle are 30°, 60° and 90°, think of two such, back-to-back, forming one equilateral △.**

Let AB be 2.

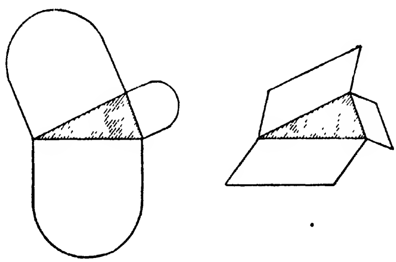


Then $\quad BC = \frac{1}{2}BX = \frac{1}{2} \times 2 = 1,$

and $\quad AC = \sqrt{AB^2 - BC^2} = \sqrt{4-1} = \sqrt{3};$

and so we get the accompanying figure; if the short side is produced we get the angle **120°**, and if the long side is produced we get the angle **150°**.

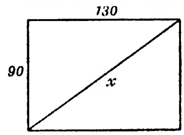
* *§ 10. **Note on areas of similar figures on the sides of right-angled triangles.** It may be noted here that the area of any figure on the hypotenuse of a right-angled triangle is equal to the sum of the areas of similar, and similarly situated, figures on the two sides. The proof only for the cases of squares is given here. The extension, to other figures besides, comes later.



• § 11. Trigonometry is based on Pythagoras' Theorem, but when one angle of a triangle is a right angle and two sides are

known, one finds the third side by using this Theorem direct, as the following examples show:

(1) A football ground is 130 yds. by 90 yds. What is the length of its diagonal to the nearest yard?



We have $x^2 = 130^2 + 90^2$

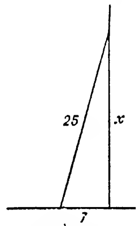
$= 16900 + 8100$

$= 25000$;

$\therefore\ x = \sqrt{25000} = 158$ about.

So that the diagonal is about 158 yards.

(2) A ladder is 25 ft. long and its foot is 7 ft. from a wall, against which it leans. How high up the wall does it reach?



We have $x^2 + 7^2 = 25^2$;

$\therefore\ x^2 = 25^2 - 7^2$

$= 625 - 49$

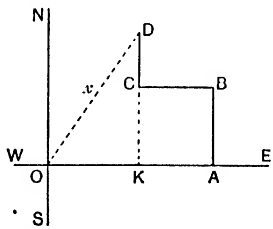
$= 576$;

$\therefore\ x = \sqrt{576} = 24.$

So that the ladder reaches 24 ft. up the wall.

(3) A man goes 14 miles East, then 10 miles North, then 6 miles West, and finally 5 miles again North. (Each lap is in continuation of the preceding, and he does not return to his starting-point every time.) How far direct is he finally from his starting-point?

The course is OABCD.



[*N.B.*—The figure is purposely not to scale; quite accurate calculations can be made from a drawing of the right sort of appearance.]

DC produced gives the right-angled triangle ODK.

The total Easting = OK = 14 − 6 miles = 8 miles.

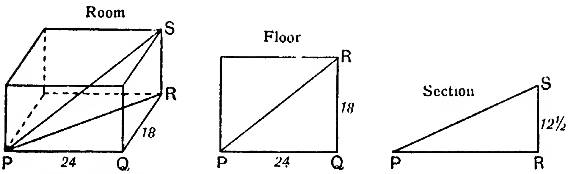
The total Northing = KD = 10 + 5 miles = 15 miles.

We have
$$x^2 = 8^2 + 15^2$$
$$= 64 + 225$$
$$= 289;$$
$$\therefore\ x = \sqrt{289} = 17.$$

And the distance direct, from start to finish = 17 miles.

* § **12.** If a figure is in 3 dimensions, it may be necessary to draw an appropriate **section.** The following example illustrates this:

(4) A room is 24 ft. long, 18 ft. wide and 12 ft. 6 in. high. How far through the air must a fly travel to go from one corner of the floor to the opposite corner of the ceiling?

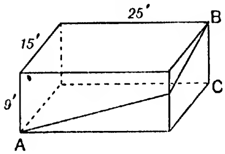


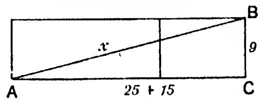
From the floor diagram
$$PR^2 = 24^2 + 18^2,$$
and from the section diagram
$$PS^2 = PR^2 + RS^2$$
$$= 24^2 + 18^2 + 12\tfrac{1}{2}^2$$
$$= 576 + 324 + \frac{625}{4}$$
$$= \frac{4225}{4};$$
$$\therefore\ PS = \sqrt{\frac{4225}{4}} = \frac{65}{2} = 32\tfrac{1}{2}.$$

So the fly has to travel 32 ft. 6 in.

** § **13.** If a distance, in spite of being direct, involves a turn we may still get a right-angled triangle. On principle **flatten it out.** The next example illustrates this idea.

(5) A spider crawls from A to B by the shortest route up two walls. How long is the route?





From the flattened-out walls diagram $AC = 40$ ft.

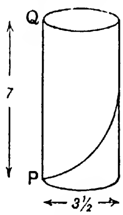
We have $$x^2 = 40^2 + 9^2$$ $$= 1681;$$ $$\therefore\ x = \sqrt{1681} = 41,$$

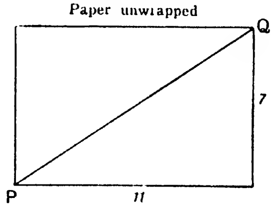
so that the shortest distance, *via* the walls, is 41 ft.

(Of course there are other possibilities too, *e.g.* *via* a big wall and the ceiling, or *via* a little wall and the ceiling. The distances in these cases are different to each other, and in neither case 41 ft. But floor and ceiling were ruled out by the conditions of the question.)

A final example will show that the "flattening-out" principle may be continuous.

(6) A cylindrical vessel is $3\frac{1}{2}$ inches in diameter and 7 inches high. A string goes from some point on the





edge of the base to the point on the top rim vertically above it, going round the vessel spirally once and being stretched tight. How long is the string? ($\pi = \frac{22}{7}$).

Imagine a piece of paper was put round the vessel first and that the position of the string, throughout its length, was marked on it; and, afterwards, that the paper was unwrapped and flattened-out. On the flattened-out diagram the course of the string (because it was stretched tight round the vessel) would be a straight line.

Now the circumference of the vessel $= \frac{22}{7} \times 3\frac{1}{2}$ in.

$$= 11 \text{ in.},$$

so that the length of the paper = 11 in.

We have

$$PQ^2 = 11^2 + 7^2$$
$$= 121 + 49$$
$$= 170;$$
$$\therefore PQ = \sqrt{170} = 13+,$$

so that the length of the string is somewhat over 13 inches.

EXAMPLES 15 a (CALCULATIONS).

One Triangle.

1. Calculate the length of the diagonal of the floor of a room, 15 ft. long and 8 ft. wide.

2. A football ground is 120 yards long and 80 yards broad. Calculate the length of its diagonal, to the nearest yard.

3. A wall is 24 ft. long and 7 ft. high. Draw it, using $\frac{1}{10}$ of an inch to represent a foot.

Calculate the distance, in feet, from the top of one end to the bottom of the other.

4. A wire is stretched right across a street 24 feet wide, and is fastened to the opposite walls at heights of 10 feet and 17 feet. Calculate the length of the wire, assuming it to be quite straight.

5. Two upright sticks are 12 ft. apart. The shorter stick is 6 ft. high and the taller is 11 ft. high. Calculate the distance between their tops.

6. Two upright sticks are d ft. apart, and the shorter stick is x ft. high and the longer is y ft. high. What is the distance between their tops?

[Hint. Work precisely as in the preceding example.]

7. A is a lighthouse. B and C are two ships 7 miles apart. B is due south of A, C due west of B, and C south-west of A. Calculate the distance of each ship from the lighthouse.

8. If the diagonals of a rhombus are 3″ and 1·6″, draw the figure and *calculate* (i) the lengths of its sides and (ii) its area.

9. The hypotenuse of a right-angled triangle is 61″ and one side is 11″. Find the area of the triangle.

10. The hypotenuse of a right-angled triangle is 6·5 cm. long, and one side is 6 cm. Calculate the third side and the area of the triangle. Also calculate the length of the perpendicular from the right angle to the hypotenuse.

11. A 41 foot long ladder leans against a vertical wall, and the bottom of the ladder is 9 ft. away from the wall. How high up the wall does the ladder reach ?

12. Find the length of one side of a rectangle if the diagonal is 2·6 inches and the other side 1 inch.

13. The diagonal of a rectangular field is 530 yards long and one side is 280 yards long. What is the length of the other side ? Also calculate the area of the field.

14. PQR is a triangle having a right angle at R. PQ = 120′, PR = 35′. Calculate the length of QR and verify by a drawing to scale. What is the area of the triangle ?

15. Two triangles have sides 20 ins., 21 ins., 29 ins., and 12 ins., 35 ins., 37 ins. Prove by calculation :

(i) that they are both right-angled triangles,
(ii) that they both have the same area.

Calculate the area of each triangle.

16. The perpendicular from the middle point of the hypotenuse of a right-angled triangle on one of the sides is three yards long, and the hypotenuse is eight yards long. Calculate the lengths of the two sides to the nearest half inch, explaining your reasoning clearly.

*** 17.** Can a brick with sides 9·5″, 4·4″ and 3·3″ pass down a cylindrical pipe whose internal diameter is 5·6″ ?

*** 18.** A man fields "point," standing quite "square" with the wicket ; at one end he is 7 yards and at the other end 8 yards from the middle stump. How far has he to walk between the overs ?

*19. A is an island, 20 miles from the nearest point, N, on the mainland, which has a long straight shore XNCBY. C is a port 21 miles from N, and B a seaside town 39 miles from N. A straight railway connects C and B. A man travels from A to C by steamer and from C to B by rail. The boat goes 20 miles an hour and the train 45 miles an hour. The train starts 25 mins. after the arrival of the boat. How long does the journey from A to B take ?

**20. Two roads cross at right angles. A and B, one on each of the two roads, start at distances 7 and 9 miles respectively from the crossing to walk towards it (A walking half as fast again as B). How far must each go so that the shortest distance between them is reduced to $6\frac{1}{2}$ miles ?

[HINT. Use Algebra. Let A go $3x$ miles and B $2x$ miles, etc.]

**21. ABC is a right-angled triangle with $AB = 7''$, $BC = 24''$, $B = 90°$.

Any straight line DE is drawn parallel to BC cutting AB, AC at D and E ; EF is parallel to AB and cuts BC in F.

Giving AD the value x'', calculate the lengths of DB, BF, FC, CE, EA and the areas of the 3 parts into which the triangle ABC is now divided.

Draw a graph to show how the area of the rectangle DEFB varies with AD.

When the rectangle is a maximum, where are D, E, F ? and what is the length of AD when the area is exactly one-fifth of the whole triangle ?

More than one triangle.

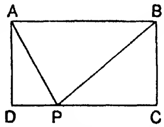
*22. P, Q, R and S are 4 points in one straight line on the ground, such that $PQ = 10$ feet, $QR = RS = 12$ feet, and QX and RY are two upright poles 24 feet and 33 feet high respectively. Find the total length of a stretched wire through P, X, Y, S.

*23. A flagstaff 40 feet high is kept upright by taut wires 41 ft. long attached to the top and to points on the ground. How far are these points from the foot of the flagstaff ?

If the same wires are attached to a point 3 feet from the top, how far (to the nearest inch) from the foot of the flagstaff on the ground will the other ends of the wires come ?

*24. A ladder has to have its foot placed 5 ft. from a wall in order to reach 20 ft. up the wall. Find to the nearest inch

how high it will reach up the wall when the foot is 3 feet from the wall.



*25. ABCD is a rectangle, 21 cm. by 8 cm., and P is some point in the line CD. Suppose CP $= x$ cm. Calculate the values of x so that $AP^2 + PB^2 = 389$ cm^2.

*26. There are three posts, P, Q, R, standing upright, in a straight line, on level ground. P is 9 ft. high, Q is 15 ft. high, R is 10 ft. high. From P to Q is 8 ft. horizontally and from Q to R 12 ft. horizontally. A wire is stretched from the top of P, over the top of Q, to the top of R. Calculate the length of the wire.

*27. When a ladder is placed with its foot 16 ft. from a vertical wall it reaches 30 ft. up the wall. How high will it reach if its foot be 10 ft. from the wall ?

*28. A ladder 27 feet long rests against a wall with 5 feet projecting over the top ; when the foot of the ladder is pushed 3 feet nearer the wall, 7 feet project over the top. Find the height of the wall.

*29. A ladder 50 feet long stands in a street, leaning against a house, where it reaches a height of 48 feet ; how far out from the house is its foot ? If the street is 30 feet wide and it is turned over to lean against the opposite house, how high will it reach ? Give your answer correct to the nearest half inch.

*30. Calculate the area of a quadrilateral field ABCD (in hectares) from the following data :

AB $= 600$ m., BC $= 250$ m., CD $= 390$ m., DA $= 520$ m., and the diagonal AC $= 650$ m.

*31. Show how to construct a rhombus with one diagonal 10 cm., and with sides 7 cm. Calculate the length of the other diagonal.

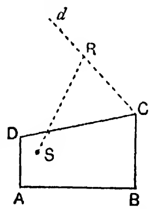
*32. Draw two triangles with sides 3, 4 and 5 cm. each, having an equal side common to both.

Draw and name the figures you obtain by varying and reversing the common side.

*33. ABC is an acute-angled triangle, in which AB $= 5''$. The projections of AB and AC on BC are $4''$ and $3''$ respectively.

Calculate one of the angles of the triangle and measure the others.

* **34.** ABCD (which is not drawn to scale) is the end view of a desk. The lid DC is hinged at C and can be kept open by a strip of wood SR which is pivoted at S, S being 1 inch below D and 1 inch to the right from AD. AD = 6 in., AB = 17 in., BC = 10 in., SR = 16 in. Make a correct drawing of ABCD,



Measure the angle BCD and deduce the angle of slope of DC to the horizontal. Find the length of DC by measurement and also by calculation.

Imagine the desk opened and the lid propped as wide open as possible by SR. Draw the figure to scale showing this position, and state your construction. What is the length of CR in this position?

* **35.** ABCD is a four-sided figure with AB = 60″, AD = 25″, CD = 52″, while the angles at A and C are right angles: calculate the length of BC.

* **36.** ABCD is a quadrilateral with AB 14·4 inches, AD 4·2 inches, CD 12 inches, while the angles at A and C are right angles. Calculate the length of BC, and the area of the figure ABCD.

* **37.** ABCD is a quadrilateral in which AB = 52″, BC = 39″, CD = 60″, DA = 25″, and the angle ABC = 90°. Calculate the length of AC; and prove that the angle ADC is also a right angle. Check your results from a figure drawn to scale.

* **38.** ABC, ADC are right-angled triangles on opposite sides of their hypotenuse AC, which is 2·6″ long.

AB = BC and AD = 1″.

Find by calculation the lengths of AB, BC, CD, and of the perpendiculars BM, DN dropped from B and D on to AC. What is the value of AN : NM : MC ?

Check by drawing a figure.

Northing, Easting, etc.

* **39.** A man travels 1 mile North, then 2 miles East, then 3 miles South, then 4 miles West, and so on till he finally goes 20 miles West. What are the total "Southing" and "Westing"? Calculate his direct distance from his starting-point.

* **40.** A man walks 10 mi. East, then 5 mi. South, then 11 mi. East, and finally 15 mi. South. Draw a sketch diagram. Complete the figure to show a large right-angled triangle, and calculate the direct distance between start and finish.

* **41.** A man walks 3 miles East, then 2·7 miles North, then $4\frac{1}{2}$ West, then 3·3 South. Draw a plan of his walk, and find how far he is from the starting-point.

* **42.** A man walks 1 mi. E., then 2 mi. N., then 3 mi. W. and finally 4 mi. S. Sketch a diagram of his course, and calculate the distance between start and finish to the nearest tenth of a mile.

* **43.** A man goes a miles E., then b miles S., then c miles W. and finally d miles N. What is the direct distance between start and finish ?

Trapezia.

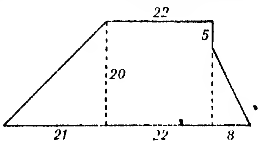
* **44.** A trapezium has its parallel sides 12 cm. and 8 cm. and each of its slope sides 6 cm. What are its height and area ?

* **45.** A railway cutting is 24 ft. deep, 16 ft. wide at the bottom, 52 ft. wide at the top. Draw a figure of this, explaining exactly how you draw each line. The slopes are the same on both sides. Calculate the length of a slanting side and the area of the cutting.

* **46.** A camping ground is in the form of a trapezium. Its parallel sides are 200 and 340 yards, and its other sides 150 and 130 yards. Find its area in acres.

* **47.** A trapezium has one of its parallel sides 2 cm. longer than the other, while the perpendicular distance between them is 3 cm. less than the shorter parallel side. The area of the trapezium is 45 cm². Calculate the length of the shorter parallel side; and, if one of the non-parallel sides be at right angles to each of the parallels, draw the trapezium correctly. Calculate the length of the longer diagonal. [Measurement on the correct figure will afford a useful check.]

* **48.** By *calculation* determine the perimeter of the figure, a rough sketch of which is given. All the lengths are in centimetres, and the dotted lines are perpendicular to the base and to the side opposite to the base. Also calculate the area of the figure.



*49. A trapezium has parallel sides 4·1″ and 2″ and the other sides 2·8″ and 3·5″ in length. Construct the trapezium by first constructing a triangle with sides 2·1″, 2·8″ and 3·5″. Show that this triangle is right angled, and hence find the area of the trapezium.

Isosceles Triangles.

50. Find the area of an isosceles triangle with sides 6·5, 6·5 and 5 inches.

51. Find the area of an isosceles triangle with sides 7, 7 and 6 inches.

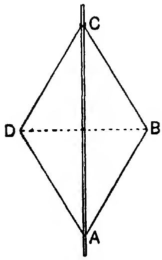
52. An isosceles triangle has its equal sides each 3·4″ and a base of 6″ : calculate its height and area.

53. Draw an isosceles triangle with base 6 cm. and equal sides 8 cm. Calculate the altitude to nearest mm., and find the area in sq. cm.

*54. Calculate the area of a right-angled isosceles triangle whose perimeter is $2p$.

*55. If c be the base, a one of the equal sides of an isosceles triangle, prove that the perpendicular from one of the angles at the base on the opposite side is $\frac{c}{2a}\sqrt{4a^2-c^2}$.

**56. AB, BC, CD, DA are four equal rods, each 5′ long, and jointed at their ends. Two rods are fixed to a vertical bar at A; the other two rods are joined to a ring C, which slides up and down the bar, C being above A. What is the figure ABCD formed by the four rods? And what do you notice about its diagonals AC and BD?



If AC $=2x$ feet and BD $=2h$ feet, express the area (y) of the figure ABCD in terms of x and h.

Find h (to the nearest tenth of a foot) when x equals 1, 2, 3, 4, 5. Tabulate values of x, h and y.

Draw a graph showing how the area y changes with x.

What is the greatest area of the figure ABCD, and what do you notice about the figure in this case?

Angles 30°, 45°, 60°, 90°, 120°, 135°, 150°.

*57. If a triangle has angles 30°, 60° and 90° and hypotenuse 12 cm., calculate its other sides and area.

*58. The side of an equilateral triangle is 2″. Calculate its altitude.

*59. ABC is an equilateral triangle, and CB is produced to E, making $EB = \frac{1}{2}BC$. Prove $AE^2 = \frac{7}{4}AB^2$.

*60. An equilateral triangle has sides $2x$ inches. Show that its area is $x^2\sqrt{3}$ square inches.

**61. A triangle has two sides 10 cm. and 14 cm. and contained angle 120°. Calculate its area.

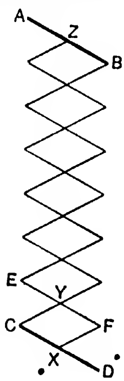
**62. A triangle has two sides 10 cm. and 12 cm. and contained angle 150°. Calculate its longest side.

*63. A triangle has one angle 135°, and the sides containing that angle 4 in. and 6 in. What is its area?

**64. There are two vertical columns: from a point midway between them their altitudes are respectively 60° and 30°. Show that one column is three times as high as the other.

**65. ABC is a triangle in which B = 30°, C = 90° and AB = 10 cm. CD is drawn perpendicular to AB, then DE perpendicular to BC, and lastly EF perpendicular to AB. Calculate the length of BF.

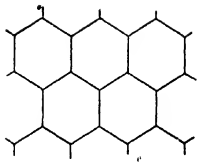
*66. The rough sketch gives a side view of a collapsible periscope. It is jointed where the bars meet or cross, and all the parts such as XC, XF, CY, FY, YE, BZ are equal in length. AB and CD are the mirrors. Prove that any quadrilateral with its opposite sides equal must also have its opposite sides parallel, and deduce that for all positions of the framework of the periscope the mirrors AB, CD are parallel.



Each of the lengths XC, XF, etc., is 3 inches. Calculate the distance of Z from X for the position in which the angle X is 90°. If a similar periscope of larger size were required, what increase in the total distance XZ would there be (in the position in which the angle X is 90°) for every inch added to the lengths XF, FY, etc.?

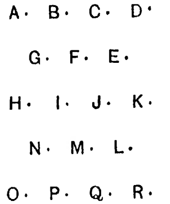
State briefly what advantages such a mechanism may have, and what drawbacks must be kept in view in its construction.

* **67.** Some wire netting is of the accompanying pattern, all the regular hexagons having sides 1 inch. Calculate the area of one of the hexagons.



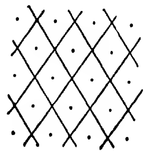
A roll of this netting is 50 yds. long and 4 ft. wide. About how many hexagons are there ?

** **68.** Apple trees are planted 5 yards apart as in the diagram. (*e.g.* CEF is an equilateral triangle with sides 5 yards.



(*a*) On unruled paper draw a correct diagram, to scale, of the position, of the 18 trees, and by each write its letter of identification. Call this diagram 1.

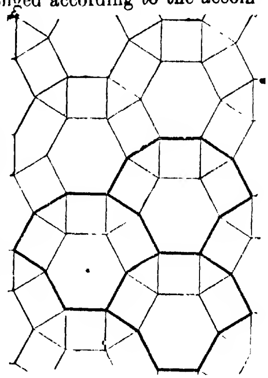
(*b*) Prick through the positions on to a separate sheet of paper and omit the letters. By ruling straight lines, show that each tree may be considered as situated at the middle of a rhombus. Call this diagram 2.



(*c*) Determine the area of one of these rhombi.

(*d*) Hence calculate the average number of trees per acre.

** **69.** The whole of a courtyard, whose area is 200 sq. yds. is to be paved with tiles arranged according to the accompanying pattern. The length of the edge of each tile is 4 in. Paying no attention to the fact that some of the tiles may have to be cut to fit the edges of the courtyard, determine how many tiles would be needed. How many would have to be hexagonal, square, triangular respectively? If the tiles can be purchased only in hundreds, how many hundreds of each must be ordered?



[Hint. Note that the whole area may be considered as made up of pieces shaped like scallop-shells.]

Triangle with given sides.

**70. ABC is a triangle with base BC 21 inches
and side CA 20 inches
and side AB 13 inches.

(*a*) Draw the triangle, on a suitable scale, and construct AL the perpendicular from A to BC.

(*b*) Let AL $=h$ and CL $=x$, then LB $=21-x$. Use Pythagoras' Theorem for the triangles CLA and ALB to obtain two equations.

(*c*) Solve these equations, determining first x and secondly h.

(*d*) Check the value of h by direct measurement on the figure.

(*e*) Knowing AL and BC, calculate the area of the triangle ABC.

(*f*) Construct BM perpendicular to CA.

(*g*) Considering CA as the base, let BM $=h'$ and AM $=x'$, then MC $=20-x'$.

(*h*) As before, write down equations to determine x' and h', and solve them.

(*i*) Check the value of h' by direct measurement on the figure.

(*j*) Knowing BM and CA, calculate the area of the triangle ABC, (check by comparing with answer to part *e*).

(*N.B.*—Of course the area could be also found by calculating the length of CN, the perpendicular from C to AB, but you are not asked to do this.)

Trigonometry Books give

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

(where $s=$semiperimeter of the triangle whose sides are a, b and c) as a formula for the area of a triangle.

Taking $a=21$, $b=20$ and $c=13$, find s; and, by substitution in the formula, obtain the area of the same triangle. (Check by comparing with answers of sections (*e*) and (*j*).)

**71. Find the area of the two parts into which a triangle, whose sides are 14, 12 and 7 cm. respectively, is divided by a perpendicular upon its longest side from the opposite angle.

* * **72.** On a base of 5·6 in. draw a triangle whose other sides are 3·9 and 2·5 in. Calculate its altitude (call the altitude h, and one part of the base x), checking your result by measurement: hence calculate its area. Find its area also by drawing and measuring a perpendicular from the smallest angle to the opposite side.

3 dimensions. Draw sections.

* * **73.** A tank stands on a base ABCD, AB = CD = 20 feet, BC = AD = 10 feet. The tank is 8 feet high and the corner above A is E.

Find the length of the diagonal AC by calculation; draw the triangle ACE correct to scale, measure CE and verify its length by calculation.

* * **74.** A and B are two opposite corners of the floor of a room and C is the corner of the ceiling above B.

If the room is 20 feet long, 15 feet broad and 11 feet high, find the length of AB, and hence of AC, correct to the nearest inch.

* * **75.** A skeleton cubic metre is made of iron rods, and a string stretches from one corner to the diagonally opposite corner. Calculate the length of the string to the nearest cm.

* * **76.** Two lines AB, AC are drawn up the walls of a room from a corner A of the floor, and meet the ceiling at points B and C 5′ and 9′ from the corner of the ceiling (above A); the line BC is drawn on the ceiling. The height of the room is 12′. Calculate the lengths AB, AC and BC, and find, by drawing, the angle BAC.

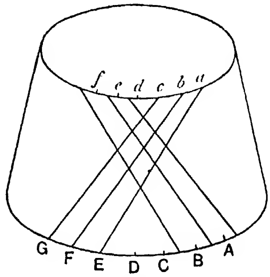
* * **77.** The floor of a room is a ft. long, b ft. wide. Calculate the diagonal of the floor. If the room is c ft. high, also calculate the diagonal of the room.

* * **78.** A long piece of wood, like a ruler, has a square section with sides 2·5 cm. and length 30 cm., but each end is finished off with a (square) pyramid whose heights are each 3 cm. (over and above the 30 cm.). Find the total volume of the wood and the total area of its surface.

* * **79.** An electric wire goes from switch to bulb; first 7 feet straight up the wall to the ceiling, then 5 feet along the junction of the wall and the ceiling, now at right angles on the ceiling for 10 feet to the middle of the room, and it hangs down for 4 feet to the bulb. How far is it (i) by the wire and (ii) direct from switch to bulb?

80. If holes are bored through the brick (diagram in question 82) direct from X to Y, Y to Z and Z to X, how long is each ?

81. A lamp shade is ornamented by 48 straight pieces of copper wire (a to E, etc.; only a few are shown in the figure to prevent confusion). The least and greatest diameters of the shade are 14″ and 20″, and the slope height is 11″. Each circle is divided into 24 equal parts a, b, c, ... etc. and A, B, C, ... etc. a and A are in the same vertical plane through the axis. The wires are aE, bF, cG, etc., and crossing them are as many sloping in the reverse direction. What is the total length of copper wire ? [HINT. Work with the isosceles trapezium aAEe.]



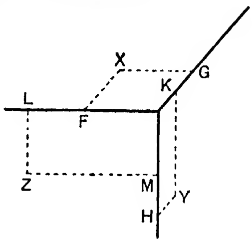
3 dimensions. "Flatten it out."

82. The diagram represents the corner of a brick.

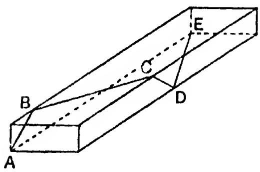
The point X is on the top face
,, ,, Y ,, ,, end ,,
,, ,, Z ,, ,, front ,,

XF = 0·5″, XG = 1·1″
YH = 0·2″, YK = 1·3″
ZL = 0·9″, ZM = 1·9″.

An elastic band goes tightly round pegs at X, Y, Z on the surface of the brick. Calculate the lengths of the 3 portions.



83. The diagram represents a block of wood 10″ by 2″ by 0·7″. A string ABCDE is stretched tightly from A to E, going once round the block. How long is the string ?



** **84.** In the previous example, if the string was stretched from A to E, going twice round the block, what is its length?

** **85.** A string, with ivy leaves, goes spirally up a cylindrical column, decorating it. The column is 11 ft. high and 1 ft. in diameter. The spiral encompasses the column twice. How long is the string?

** **86.**

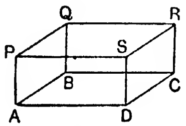


Fig. i.

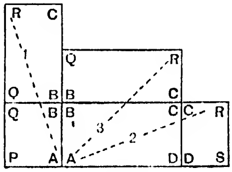


Fig. ii.

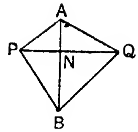
In Fig. i. ABCD is the floor of a room.
PQRS is the ceiling.
The length AD of the room $= a$ feet.
The breadth AB of the room $= b$ feet.
The height AP of the room $= c$ feet.

In Fig. ii. three of the walls are imagined hinged along the floor-edge and bent flat in the same plane as the floor, while the wall QBCR is *also* drawn as hinged along QB.

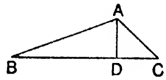
A spider and a fly are at A. The latter escapes from the former and wings her flight direct to R. The spider crawls diagonally up the walls ABQP and QBCR (by route 1) to the same goal. On arrival at R the fly turns instantaneously and returns towards A direct. The spider reaches R, and seeing the fly on her way back to A, pursues a different course to A, diagonally down the wall CDSR and across the floor (by route 2); only to find on arrival at A that the fly had reached that point just previously, and had again made off for R. The spider trekked across the floor diagonally and up the wall RCBQ (by route 3), and arrived simultaneously with the fly at R, where she fell a victim to his perseverance.

She flew at k feet a second; how fast did he crawl?

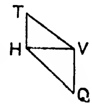
EXAMPLES 15 b (RIDERS).



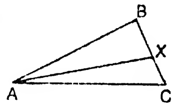
1. AB and PQ are two straight lines meeting at right angles at N.
Prove that $PB^2 + QA^2 = PA^2 + QB^2$.
[HINT. $PB^2 = PN^2 + NB^2$, etc. Add.]



2. In any triangle ABC, a perpendicular AD is drawn from A to BC. Prove that $AB^2 + CD^2 = AC^2 + BD^2$.
[HINT. Get each side equal to 3 squares.]



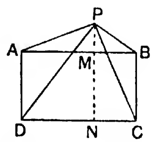
3. HVQ is a triangle, right-angled at V. HVT is a triangle on the opposite side of HV, having the angle THV a right angle. Prove that the sum of the squares on TH and HQ is equal to the sum of the squares on TV and VQ.
[HINT.—$TH^2 + HQ^2$ = what 3 squares ?]



4. ABC is a triangle, right-angled at B. In BC any point X is taken. Prove that
$$AX^2 + BC^2 = AC^2 + BX^2.$$
[HINT.—Get each side equal to 3 squares.]

5. ABC is any triangle. A point O is taken within it and OX, OY, OZ are drawn perpendicular to BC, CA, AB respectively. Prove that the sum of the squares BX, CY, AZ is equal to the sum of the squares XC, YA, ZB.

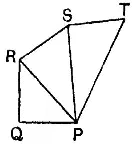
6. Draw a line ABM perpendicular to a line CMD. Prove that $AC^2 - BC^2 = AD^2 - BD^2$.



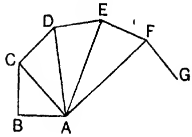
7. If ABCD is a rectangle and P any point, prove that $PA^2 + PC^2 = PB^2 + PD^2$.
[HINT.—First draw the perpendicular PMN.]

8. ABCD is a square and BD one of its diagonals. Prove the square on BD must be double the square ABCD.

9. ABC is an equilateral triangle, and AM is perpendicular to the side BC. Prove that $AM^2 = \frac{3}{4}AB^2$.



*10. In the figure $PQ = QR = RS = ST$, and there are right angles at Q, R and S. Prove that PT must be twice PQ.



*11. In the figure
$$AB = BC = CD = DE = EF, \text{ etc.},$$
and there are 25 of these in all (Z being the end). The angles at B, C, D, ... etc., are right angles. Prove that AZ must be 5 times AB.

*12. Show how to construct a square equal to twice a given square, and give proof.

*13. Find by construction the side of a square equal in area to the sum of three squares whose sides are 3, 4 and 5 cm. respectively, and check by calculation.

*14. ABC and DBC are equilateral triangles described on opposite sides of BC: join AD and *prove* that the square on AD is three times the square on AC.

*15. ABC is a triangle having B a right angle. BD is drawn perpendicular to AC, meeting AC at D. Prove

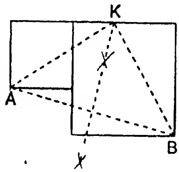
(1) that $AC^2 = AD^2 + DC^2 + 2BD^2$,

(2) that the square on BD is equal in area to the rectangle whose sides are AD and DC.

*16. A and B are two fixed points, and P is a variable point such that the difference of the squares on PA and PB is constant. Prove that P describes a straight line.

*17. The angle A of a triangle ABC is a right angle. P, Q, R are the middle points of BC, CA, AB. Prove that
$$CR^2 + BQ^2 = 5QR^2.$$

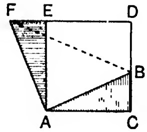
**18. The two given squares (of sides 5 cm. and 7 cm.) are to be cut into three pieces along the lines AK and BK; these three pieces are to be formed into a single square. Show carefully how the pieces must be placed together to form this single square.



The point K may be found by bisecting the line AB at right angles. Suggest, if you can, a simpler method of finding the point.

**19. The following is given as a proof of Pythagoras' Theorem:

ABC is a right-angled triangle, in which the side CA (or b) is greater than the side BC (or a).



To prove that $a^2 + b^2 = c^2$.

Describe the square ACDE.

Revolve the triangle ABC round A to the position AFE.

Proof:

Area ACDE = area ABDF [reason ?].

$\therefore$ Area ACDE = area FAB + area FBD.

$\therefore$ Area ACDE = $\frac{1}{2}$AB × FA + $\frac{1}{2}$BD × DF [reason ?].

$\therefore$ $b^2 = \frac{1}{2}c^2 + \frac{1}{2}(b-a)(b+a)$ [reason ?].

Whence by usual algebraical processes $a^2 + b^2 = c^2$.

Re-write the proof, stating the reasons clearly, and carrying out the algebraical processes fully.

**20. If the sum of the squares on two opposite sides of a quadrilateral is equal to the sum of the squares on the other two, the diagonals intersect at right angles.

**21. On the sides of a right-angled triangle ABC, squares ABDE, BCFG, CAHK are drawn. Prove that the sum of the squares on DG, FK is equal to 5 times the square on the hypotenuse BC.

(The squares are to be drawn externally.)

[Hint. Draw GS perpendicular to DB produced.

Prove (1) $\triangle$SBG $\equiv$ $\triangle$ABC,

(2) $DG^2 = (2c)^2 + b^2$, etc.]

**22. ABC is an acute-angled triangle in which the side AB is greater than BC, and D is the foot of the perpendicular from B on AC. A point E is taken in AC produced so that CE = CB, and on DE as base a triangle DPE is constructed, having its sides DP, PE equal respectively to DA, AB. Prove that PC is at right angles to DE.

CHAPTER XVI.

LOCI.

§ 1. **Definition.** A locus is the path traced out by a point which moves according to some fixed law.

[*N.B.*—The Latin word *locus* means "a place"; however, this Mathematical term does not mean a *single place*, but the *aggregate of all the places* which satisfy the fixed law.]

Now, in a particular case, the law may be simple and capable of being put into simple words, and yet the locus may be complicated mathematically, but easy to draw moderately well freehand if we obtain sufficient points to guide us. The locus of many common things, as they are compelled to move according to laws with which we are very familiar, may or may not be quite simple: the locus of the handle of a door, which is opened and shut, is quite simple; while the locus of one of our knees, as we get out of bed, may be complicated (though the law is simple enough).

For loci which are going to turn out to be bits of straight lines or bits of circles knowledge of elementary geometry is everything. Experience tells a great deal. and with ruler and compass we can draw those loci so much better than freehand.

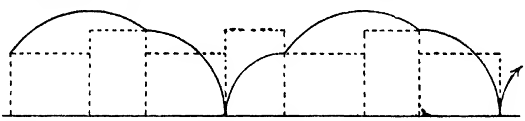
To see what kind of locus some law produces, we should try to obtain several points on the locus, and then possibly jump to conclusions; afterwards, of course, we must test our locus to see whether after all we jumped to the right conclusions. It is very easy for us to admit a certain curve, when it is shown to us, to be correct; but it is quite a different thing trying to pick holes in our own handiwork. We naturally are apt always

to think our own efforts sound, and are unwilling (and think it a mere waste of time) to try to find fault with our own conclusions; but it is the only proper way really.

Loci may turn out to be (i) bits of straight lines, bits of circles or both, or (ii) other curves. For the former our knowledge of elementary geometry comes in; and rulers and compasses should be used in the drawing. For the latter, though machines can be invented to draw the curves, they are not universally satisfactory. A series of points is plotted (tracing paper and pin-pricks are especially useful in some loci), and these points may be joined up freehand. [We can make the locus look nicer by using an appropriate "French Curve," "Ship's Curve" or "Brook's Curve," or such like judiciously, but the results are apt to be disappointing unless one practises their use.] In some cases a knowledge of higher mathematics is very useful, but by no means essential to nice figures.

§ 2. **Some examples,** illustrating various methods, are given below. The number of possibilities is really infinite; and we must gain experience for ourselves rather by working out very many examples than by reading and gazing at the labours of others.

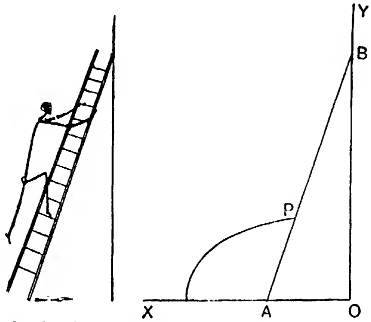
(1) The locus of one corner of a book that is rolled over and over along a straight line is the series of circular arcs shown.



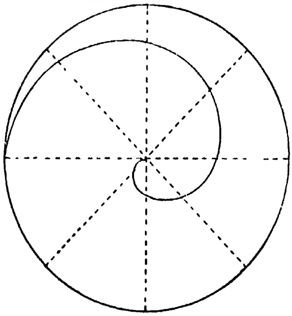
Spotting what are the successive centres and radii will make an accurate figure easy.

(2) The locus of one of the rungs of a ladder as the ladder slips is more elaborate mathematically, though very easy to draw. Suppose a man were climbing up the ladder when the ladder begins a slip. Of course the man clutches tight hold of the ladder, and his foot, on the rung, is still as far as the ladder is concerned. The essential features are shown in the second diagram. It is a case for tracing paper and pin pricks.

Draw OX and OY on plain paper, the ladder AB on tracing paper. Move the tracing paper about, but keeping A on OX and B on OY. Prick through the given point P often. The series of pin pricks joined up freehand will give the locus.



(3) The wheel of a stationary engine revolves (clockwise), while a fly crawls along a given spoke, from rim to hub, arriving when the wheel has revolved once. Both move uniformly. The locus of the fly is shown. This is a case for plotting many



points and joining them up freehand. (Eight positions of the spoke give us lines on which to mark spots $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, ... etc. way along, so as to give us points to join up freehand. With sixteen positions of the spoke, of course more accurate results can be obtained.)

§ **3.** Two locus-propositions are now formally treated.

The proof of a locus-proposition consists of two distinct parts.

(i) Proof that all the points, which satisfy the given conditions, fall on a particular line (or lines) which are called the locus (of course the lines may be straight or curved), and (ii) proof that *all* the points on this line (or lines) satisfy the given conditions (and are therefore on the locus); and, if not the whole line, what part of it.

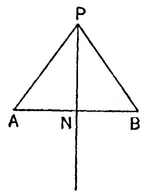
PROPOSITION 16.

§4. **General Enunciation.** *The locus of a point equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.*

Particular Enunciation. A and B are the two fixed points; the locus of a point equidistant from A and B is the perpendicular bisector of AB.

(Part i.)

Construction. Let P be a point on the locus (*i.e.* a point such that PA = PB). Join AB. Draw PN perpendicular to AB.



Proof. In the rt. ∠d △s APN, BPN,

$$\because \begin{cases} PA = PB \text{ (given)}, \\ PN \text{ is common}, \\ \angle ANP \text{ and } \angle BNP \text{ are rt. } \angle s, \end{cases}$$

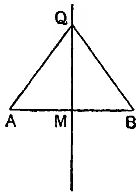
$\therefore$ $\triangle APN \equiv \triangle BPN$,

and, in particular, AN = BN.

Hence P must lie on the perpendicular bisector of AB.

(Part ii.)

Construction. Bisect AB at M. Let Q be any point on the perpendicular bisector of AB. Join QA and QB.



Proof. In the $\triangle$s AQM, BQM,

$$\because \begin{cases} AM = BM \text{ (bisection)}, \\ QM \text{ is common}, \\ \angle AMQ \text{ and } \angle BMQ \text{ are rt. } \angle s, \end{cases}$$

$$\therefore \triangle AQM \equiv \triangle BQM,$$

and, in particular, QA = QB.

Hence Q (*any* point on the perpendicular bisector, and therefore *all* points thereon) is equidistant from A and B.

Conclusion. So that the *whole* perpendicular bisector of AB forms the locus. **Q.E.D.**

N.B.—Parts i. and ii. are different. In (i) it is shown that points on the locus (and P is one of these) are confined to the perpendicular bisector; and in (ii) that any point on that perpendicular bisector (and Q is one of these) is equidistant from A and B. You conclude that the locus is the *whole* perpendicular bisector.

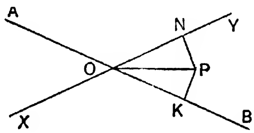
PROPOSITION 17.

§ 5. **General Enunciation.** *The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.*

Particular Enunciation. AB and XY are the two straight lines intersecting at O. The locus of a point equidistant from AB and XY consists of the pair of straight lines bisecting the angles at O.

(Part i.)

Construction. Let P be a point on the locus (*i.e.* a point such that the perpendiculars from it to AB and XY are equal). PK is perpendicular to AB, and PN to XY. Join OP.



Proof. In the right $\angle$d $\triangle$s OPN, OPK,

$$\because \begin{cases} PN = PK \text{ (given)}, \\ OP \text{ is common}, \\ \angle PNO \text{ and } \angle PKO \text{ are rt. } \angle s, \end{cases}$$

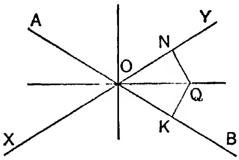
$$\therefore \triangle OPN \equiv \triangle OPK,$$

and, in particular, $\angle PON = \angle POK$.

Hence P must lie on one or other of the bisectors of the angles at O.

(Part ii.)

Construction. Bisect the angles at O. Let Q be any point on one of these bisectors. QK is perpendicular to AB, and QN to XY.



Proof. In the $\triangle$s OQN, OQK,

$$\because \begin{cases} \angle QON = \angle QOK \text{ (bisection)}, \\ \angle QNO \text{ and } \angle QKO \text{ are rt. } \angle\text{s}, \\ OQ \text{ is common}, \end{cases}$$

$$\therefore \triangle OQN \equiv \triangle OQK,$$

and, in particular, QN = QK.

Hence Q (*any* point on one of the bisectors of the angles, and therefore *all* points thereon) is equidistant from AB and XY.

Conclusion. So that the *whole* of the bisectors of both angles form the locus. **Q.E.D.**

N.B.—You have to notice that Parts i. and ii. are different. In (i) it is shown that points on the locus (and P is one of these) are confined to the bisectors of the angles at O; and in (ii) that any point on these bisectors (and Q is one of these) is equidistant from AB and XY. You conclude that the locus is the ***whole*** of the bisectors of both angles.

EXAMPLES 16 a (GEOMETRICAL DRAWING).

1. Draw a triangle ABC having the angle $A = 36°$, $BC = 2$ inches and $AB = AC$. On AB find a point P equidistant from BC and CA. Show your construction quite clearly and measure this perpendicular distance.

2. Draw a triangle ABC, given $\angle B = 36°$, $\angle C = 76°$, $BC = 3{\cdot}5''$.
Find two points equidistant from AB and BC, and $2{\cdot}1''$ from C. Measure their distance apart. Explain your construction.

3. Construct a parallelogram whose sides are $5''$ and $3''$, and the distance between the longer sides $2{\cdot}5''$.
Find the points in the longer sides
(1) equidistant from the ends of one diagonal,
(2) equidistant from the ends of the other diagonal.
Measure the distance between the points in each side.

4. Draw two straight lines, POQ, ROS, intersecting at O at an angle of 37°. Find a point X whose perpendicular distance from each of these lines is 1·5 cm., and measure OX. If possible, find more than one position of X.

5. ABC is a triangle with $AB = 3''$, $BC = 2{\cdot}5''$, $CA = 1{\cdot}3''$. Find a point P, $0{\cdot}6''$ from BC and equidistant from AB and AC.
Explain your construction carefully.
Measure the distance of P from AB.

* **6.** I have two friends living at two villages, A and B, which are three miles apart. A road runs through A, making an angle of 35° with AB. I want to build a house half a mile from this road, and equidistant from my friends. Where must I build it? (Explain how you find the site.) Are there two possibilities? Measure the distance from the house to either village.

* **7.** Draw a circle with radius $1\frac{1}{2}$ in., and a line $\frac{1}{2}$ in. from its centre. Describe a circle of 1 in. radius to touch both. How many solutions are there?

* **8.** Construct a triangle ABC having $AB = 10$ cms., $AC = 9{\cdot}3$ cms. and perpendicular from C to $AB = 8$ cms. State your construction very briefly and measure CB.

* **9.** Draw two straight lines OA, OB inclined at an angle of 60°. Find a point P which is $2''$ from OB and $1''$ from OA, and a point Q $\cdot 5''$ from OA and $1{\cdot}5''$ from OB. Find also one or

two other points which are an inch further from OB than they are from OA. What is the locus of these points? Justify your results by general reasoning.

[The lines OA and OB are not to be produced, backwards, beyond O.]

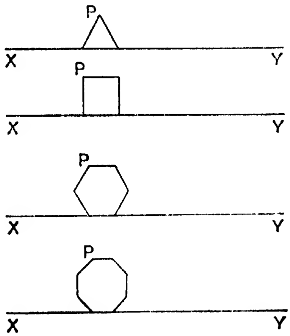
*10. An equilateral triangle, a square, a regular hexagon and a regular octagon are rolled along separate straight lines XY, without slipping. Draw the locus of one of the corners P of each for a complete revolution.

N.B.—The side of the equilateral triangle and of the square may be taken as 5 cm.

For the hexagon and octagon first draw circles with radii 3 cm. in which to place these figures. These circles and figures should be drawn originally on unruled paper distinct from that on which the loci are shown. (It may help you if you draw them on tracing paper.)

Give a sketch (an accurate figure is not required) of the locus of a speck of mud on the rim of a cartwheel, which is rolling along a straight, flat road.

Do you or do you not imagine that the locus is an arc of a circle?

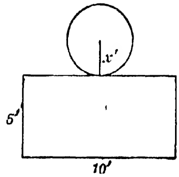


*11. A halfpenny, whose diameter is 1″, rolls first round the outside and then round the inside of an equilateral triangle whose sides are 4·8″. Show the loci of its centre.

*12. An equilateral triangle, with sides 2·5 cm., rolls round the outside, and also round the inside, of a rectangle whose

sides are 12·5 cm. and 7·5 cm. Show the loci of one of the angular points of the equilateral triangle.

* **13.** A circle of radius x feet rolls completely round a rectangle 10 ft. by 5 ft. as in the rough sketch. Trace the path of the centre of the circle.

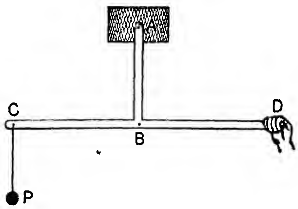


* **14.** ABCD is a kite-shaped figure; ABC being an equilateral triangle of side 5·6 cm., and ADC a semicircle of radius 2·8 cm.

AB is in contact with a straight line XY. Draw the locus traced out by B, as the figure ABCD rolls without slipping along a straight line XY, and completes one revolution. How far, along XY, has B moved when it lies on XY again? Give an accurate check.

ABCD may be drawn on tracing-paper, or cut out on ordinary paper.

* **15.** $\begin{cases} AB = 30 \text{ cm.} \\ BC = 40 \text{ cm.} \\ CP = 20 \text{ cm.} \end{cases}$



AB, CD are two rods at right angles to each other and rigidly connected at B.

P is a heavy ball connected to the point C by string.

The whole system can rotate about a nail at A.

Find the locus of P when the rotation is so slow that P is always vertically under C.

Draw on a scale of 1 : 10, and only show positions of P below the horizontal line through A, and to the left of the vertical line through A.

[You may cut a model out of paper, if it will help you.]

* **16.** ABC is a triangle; BA is produced to D so that AD = AC; AK is drawn parallel to DC to meet BC in K. Prove AK bisects the angle BAC.

Draw a figure in which AB is 8 cm. and AC is 5 cm. Keeping AB fixed, show various positions of AC according as the angle BAC is 20°, 50°, 90°, 120° and 150°. Obtain the various positions of K by drawing lines through A parallel to DC.

Mark these points K very clearly, and draw (freehand) the locus of K.

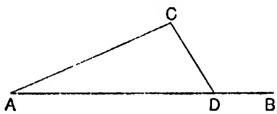
If you find that the locus of K is a semi-circle, state what you think is its radius.

Preferably with tracing paper and pin pricks.

**17. One end of a rod 12 cm. long moves on a circle of 4 cm. radius. The other end moves on a straight line which passes through the centre of the circle. Plot the locus of the middle point of the rod.

What is the distance between the extreme points of the path ?

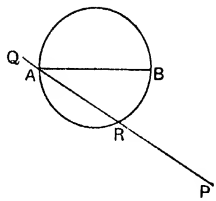
**18. The rough sketch represents part of a machine. The rod AC can revolve freely about the point A; the rod DC is freely jointed to AC at C, and D moves on a slot in AB, the whole motion being in one plane. If AC = 24 in. and CD = 8 in., find by drawing the range of movement possible for AC and the point D, giving for AC the total angle it can turn through, and for D its total movement. Draw a figure to illustrate that if the angle CAB is 10° there are two possible positions for D, and measure their distance apart.



[AB is in a fixed position.]

**19. Two straight lines XX′ and YY′ are given cutting one another at right angles. A and B, fixed points on a given straight line BAP, are compelled to move respectively on them. Draw the locus of P. [BA = 4 cm. and AP = 3 cm.]

**20. AB is a fixed diameter of a circle. A variable straight line ARP is compelled to pass through A and it cuts the circle at R. From R is measured RP or RQ along the line and equal to the diameter of the circle. Show the loci of P and Q.



**21. Two circles with radii 3 cm. have their centres 7·5 cm. apart. A straight line, 4 cm. long, is compelled to move so that each of its extremities is on one of the circles. Show the locus of the middle point of the straight line.

Intersecting arcs.

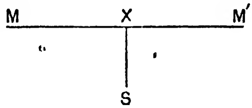
* * **22.** Define what is meant by the locus of a point. Mark two points, A, B, on your paper 2 inches apart. By plotting points, draw the locus of a point P such that

$$AP + PB = 4 \text{ inches.}$$

[First copy and fill in the following table]:

AP	0″	·5″	1″	1·5″	2″	2·5″	3″	3·5″	4″
PB									

* * **23.** Draw the locus of a point which moves so that its distance from the given point S is equal to its distance from the given straight line MM′.



[SX = 2 cm.]

[Hint. Rule parallels to MM′ at distances 1, 2, 3, etc. cm. Then with centre S and radius 3 cm. cut the 3 cm. line, etc. Similarly plot other points.]

* * **24.** A fly crawling uniformly travels along one of the spokes of a wheel which is revolving uniformly. The fly goes from rim to hub while the wheel revolves twice and then immediately crawls back at the same rate. Draw the locus of the fly.

* * **25.** A piece of string 12″ long is fixed to 2 points X, Y, which are 9″ apart; a pencil is placed inside the string and moved while the string remains tight. Draw *roughly* the locus of the pencil point. If this locus cuts the perpendicular bisector of XY at P, and the circle on XY as diameter at Q, calculate (i) the distance of P from XY, (ii) the distance of Q from XY.

EXAMPLES 16 b (RIDERS, etc.).

[*Some of these questions might be discussed orally. To such the letter* O *is prefixed.*]

O 1. What is the locus of the centre of all circles which pass through 2 given points A and C?

O **2.** How would you find the centre of the circle which touches the sides of the triangle ABC, BC externally and AB and AC internally ?

O **3.** What is the locus of the centre of all circles of radius 2 inches which pass through a given point R ?

O **4.** What is the locus of the centre of all circles of radius 2 inches, which touch a given line PQ on the same side ?

O **5.** What is the locus of the centre of all circles which touch a given line BC at C ?

O **6.** What is the locus of vertices of triangles on a given base 6 cm. long and having an area of 30 square cm. ?

O **7.** What is the locus of the vertex D of a triangle DEF drawn on a fixed base EF 10″ long so as to have an area of 20 square inches ?

O **8.** State the loci of the following :

(*a*) Points equidistant from two given points.
(*b*) The vertices of isosceles triangles on a fixed base AB.
(*c*) The centres of circles of 3″ radius drawn through a fixed point C.

O **9.** State the locus of the following :

(*a*) Points equidistant from a pair of intersecting straight lines.
(*b*) Centres of circles of radius 5 in. touching a given circle of radius 8 in.
(*c*) Vertices of triangles of area 15 square in. on a fixed base AB, 10 in. long.

O **10.** What is the locus of the centre of a circle of radius 2″ which rolls round the inside of another circle of radius 3″ ?

11. XY is a straight line 4 in. long and AX, BY (perpendiculars to XY on the same side of it) are respectively 3 in. and 2 in. long. Give the construction for finding a point P, in XY, equidistant from A and B.

12. In the previous question, if the perpendiculars AX and BY are on opposite sides of AB, solve the problem.

13. PQRS is *any* quadrilateral. Show how to find a point in RS (produced if necessary) which is equidistant from P and Q.

14. A river flowing north and south is crossed by a road going east and west ; a path is such that every point on it is

equidistant from river and road. Draw the path, and prove that every point on it satisfies the required condition.

15. A triangle has a given area and its base of given length; if the vertex be fixed, find the locus of the foot of the perpendicular from the vertex upon the base.

16. A finite straight line AB of constant length moves so that its ends A and B always lie on two fixed straight lines OX, OY at right angles to one another. Find the locus of the point of intersection of the perpendiculars through A and B to OX and OY respectively.

* **17.** OAB and OCD are two straight lines and AB is equal to CD. Show how to find a point X such that XA = XD and XB = XC. Prove that the angles AXB and CXD are equal, and that the perpendiculars from X on AB and CD are also equal.

* **18.** Draw an isosceles triangle on a given base so that the difference between one of its sides and its altitude is equal to a given straight line.

* **19.** Find the locus of the middle point of a straight line drawn from a given point to meet a fixed straight line.

* * **20.** BC is a line fixed in position and of 7 cms. length. D is a point which is always 5 cms. from B. CD is produced to A, where AD = CD. What is the locus of A?

* * **21.** From a fixed point X a perpendicular XP is drawn to a variable straight line YP, which passes through a fixed point Y. XP is produced to H, so that PH = XP. Find the locus of H.

* * **22.** Find the locus of a point the sum of whose distances from two given straight lines is fixed in length.

CHAPTER XVII.

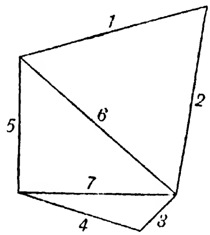
THE NUMBER OF DATA REQUIRED FOR THE DETERMINATION OF FIGURES.

(This chapter should be omitted entirely in a first reading of the subject.)

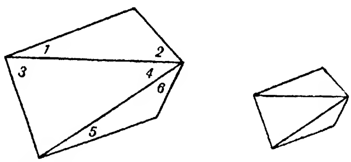
* § **1.** If the exact position of a point has to be reproduced it is necessary to make two measurements. For instance, if the position of the dot of this **i** on this page has to be reproduced on the last page of the book, we might measure how far it is from the left-hand side of the page, and also how far it is from the bottom of the page. Those two measurements are sometimes called its coordinates with regard to the edges of the page, and we are probably familiar with them in graphs. To reproduce a straight line in position we need 4 measurements (2 for each end). To reproduce a triangle in position, 6 measurements (2 for each corner), and so on. To reproduce a polygon of n sides, in **size, shape and position,** $2n$ measurements are needed.

* § **2.** Now position is everything in the rediscovery of some old buildings, the excavations of which have been filled in, but there is seldom any question as to what particular portion of our paper we use for the drawing of our figures. If we are concerned only with **size and shape** (and not position) we need 3 less facts. We do not need the position on the page, and thereby we save 2 facts, nor do we need to know at what angle the figure is tilted, and thereby we save the third fact. We can reproduce lines more accurately than

angles, and it is wise (if we have the option of choosing them) to measure lines rather than angles. Thus for a 5-sided polygon 7 measurements are needed [10 − 3 = 7]. For an n-sided figure we should need $2n - 3$ facts to reproduce it correctly in size and shape. (Of course this implies that none of our measurements have been redundant. We should not measure the third angle of a triangle, after measuring the other two, except as a check. It would not count as one of the necessary measurements.)



* § **3.** If we are only concerned with **shape**, both position and size being immaterial, one extra measurement can be saved. Thus for a pentagon 6 measurements are needed [10 − 4 = 6].



For an n-sided figure we should need $2n - 4$ facts.

* § **4.** All this of course implies the **least** that is necessary to reproduce polygons of *any* kind, but if we know what sort of polygon it is we can reproduce it with fewer given facts (of course using all the facts we know to be true of the particular sort of figure we are concerned with).

Some **examples** are appended to illustrate this :

(1) Make a square on a base of 2″. Only one fact is given.

but we know enough about squares to supply what is missing. In the first place the 3 other sides must be also 2″, and the fact that a square must be right-angled supplies a necessary fifth fact.

(2) Make a *regular* pentagon on a base 3 cm. Here we can calculate that each angle must be 108°, and because the other sides must be 3 cm. too, the fact (apparently one) was all sufficient.

Sometimes we can take the shape for granted; for instance, a carpet 15 ft. by 12 ft. must mean a rectangular carpet; the Ace of Diamonds with one side 1 cm. and one angle 60° must be in the shape of a rhombus, and our knowledge of the properties of a rhombus supplies us with the facts that are (apparently) missing.

EXAMPLES 17 (GENERAL).

*1. Rule two straight lines at right angles and reproduce, relative to them, the positions of the points P and T equivalent to their positions on the last page of the book. How many measurements are needed? [You can test your success by making a copy of the original on tracing paper.]

*2. Suppose a question began "Draw a triangle with . . . .". If it were necessary to reproduce the triangle correctly in size and shape, how many facts should be given after the word "with"? If only shape were necessary, what would be the number?

*3. How many rhombi, different in size and shape, each with diagonals 4·2″ and 3″ can be drawn?

*4. "Draw a right-angled isosceles triangle with base 3″." Why is apparently only one measurement sufficient to reproduce this triangle correctly?

*5. "Draw two straight lines to represent the rails on a railway track." How many measurements are needed if neither the gauge nor the length of the lines matters?

*6. If it is needed to reproduce correctly, relative to OX and OY, all the 11 points on the last page of the book, how many measurements are needed?

* 7. How many different rhombi (their positions are immaterial) can be drawn with sides 6 cm. ?

* 8. How many data are needed to draw a regular hexagon, if its position and size do not matter ?

* 9. How many facts are needed to draw correctly a trapezium in size and shape ?

* 10. Consider the two separate examples beginning :

(i) " Draw the quadrilateral ABCD, such that . . . ."

(ii) " Draw the parallelogram PQRS, such that . . . ."

What is the number of data required in each of the two separate cases ?

* 11. In a field an ancient square building has been filled in and there are no traces left. How many measurements from two hedges of the field should have been made to determine its position at once ?

* 12. On the same diagram draw three graphs to show the number of data required to reproduce a figure (bounded by straight lines) in (i) size, shape and position, (ii) size and shape, and (iii) shape only.

[Horizontally, 0, 1, 2, 3, 4, ... sides ; vertically, number of data.]

Your graphs will naturally start at the three sides ; and you may, wisely, join up the points by straight lines, though anything but a whole number of sides seems rather impossible. You will be able to see what the graphs might say for a two-sided figure. Do the answers seem sensible for a two-sided figure (which is of course only two coincident straight lines) ?

* 13. Draw *any* triangle ABC on a piece of paper. (1) What is the least number of measurements on ABC which must be made in order to be able to draw a similar triangle of the same *shape* ? (2) What least number of measurements for the same *shape* and *size* ? (3) What least number of measurements for the same *shape*, *size* and *position* (relative to the sheet of paper) ?

In the last case on a second sheet of paper construct a triangle XYZ exactly agreeing with ABC ; and as a check lay the first paper on the second and prick through the points A, B, C. Indicate very clearly what measurements you made.

*14. How many data are required for the construction of a quadrilateral of a particular size and shape ?

Construct the quadrilateral ABCD with AB 3·9 cm., BC 5·2 cm., $\angle$ABC 90°, CD 2·5 cm., DA 6 cm., $\angle$ADC 90°.

How many of the data are superfluous ? State which you did not use in the construction (but which you are advised to use as a check). Calculate the area of ABCD (i) as the sum of two triangles, and (ii) by reduction to a triangle.

In this particular case it is possible to *calculate* the area exactly, independently from an accurate figure. *Calculate* the exact area and give as a percentage (to one significant figure) the difference between the exact area and the area you obtain by reduction to a triangle.

*15. Fold a piece of rectangular paper so that two opposite corners coincide. Call the resulting pentagon ABCDE, A being the angle where the two corners of the original piece of paper coincide.

Draw ABCDE to a scale of 1 : 4. State what measurements you make for the construction, and also what measurements in order to check your drawing.

How many data are required for the construction of a pentagon of a particular size and shape ?

ABCDE is symmetrical about a certain line. Point out the corresponding line in your drawing, and state which and how many measurements were made unnecessary for the drawing of ABCDE owing to its symmetry.

Reduce the drawing of the pentagon to a triangle of equal area, and write down the area of ABCDE.

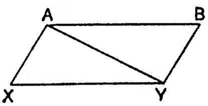
APPENDIX.

For convenience certain propositions (proofs of which were only outlined in the text) are given in full here.

PROPOSITION A.

General Enunciation. *Straight lines which join the ends of equal and parallel straight lines, towards the same parts,* must themselves be equal and parallel.*

Particular Enunciation. AB and XY are the equal and parallel straight lines. To prove that the joins BY and XA* must be equal and parallel.



Construction. Join AY.

Proof. In the $\triangle$s YBA, AXY,

$$\because \begin{cases} AB = YX \text{ (given)}, \\ AY \text{ common}, \\ \angle BAY = \angle XYA \text{ (alt.)}, \end{cases}$$

$\therefore$ $\triangle$s YBA, AXY are congruent,

and, in particular, $\begin{cases} BY = XA, & \text{..............................(i)} \\ \angle BYA = \angle XAY. \end{cases}$

Now these angles are alternate.

So that BY and XA are parallel. ....(ii)

Hence BY and XA are equal and parallel. **Q.E.D.**

[It is important to note that this makes ABYX a parallelogram.]

* "Towards the same parts" means B is joined to Y and X to A and not crosswise BX and AY.

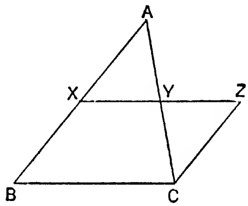
PROPOSITION B.

General Enunciation. *The join of the mid-points of two sides of a triangle must be parallel to the base and half thereof.*

Particular Enunciation. ABC is the triangle. X and Y are the mid-points of AB and AC.

To prove that XY must be parallel to BC and half of BC.

Construction. Produce XY to Z, making YZ equal to XY. Join ZC.



Proof. In the $\triangle$s CZY and AXY,

$$\because \begin{cases} YC = YA \text{ (given)}, \\ YZ = YX \text{ (constructed)}, \\ \angle CYZ = \angle AYX \text{ (vertically opposite)}, \end{cases}$$

$\therefore$ $\triangle$s CZY and AXY are congruent,

$$\text{and, in particular, } \begin{cases} ZC = XA, \text{ ....(i)} \\ \angle YZC = \angle YXA. \text{ ....(ii)} \end{cases}$$

Now XB is also equal to XA (construction),

so that $XB = ZC$;

also the fact that $\angle YXA$ and $\angle YZC$ are alternate makes AXB and ZC parallel.

Hence XB and ZC are equal and parallel.

In other words, XZCB is a parallelogram,

so that XZ is parallel to BC and equal to BC;

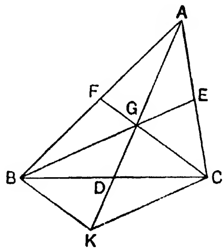
$\therefore$ XY is parallel to BC and half BC. **Q.E.D.**

PROPOSITION C.

General Enunciation. *The medians of a triangle are concurrent and trisect each other.*

Particular Enunciation. ABC is the triangle. E and F are the mid-points of AC and AB. BE and CF meet at G. To prove that AG produced bisects BC.

Construction. Produce AG to K, making GK equal to AG. Join BK and KC.



Proof. In the $\triangle$ABK the points F and G are the mid-points of the sides AB and AK ;

$\therefore$ FG is parallel to BK ; (Prop. B)

$\therefore$ FC is parallel to BK.

Similarly BE is parallel to KC.

$\therefore$ BGCK is a parallelogram, so that its diagonals BC and GK must bisect each other.

$\therefore$ GD $=\frac{1}{2}$GK (*i.e.* GD $=\frac{1}{2}$GA or AD is trisected at G),

and D is the mid-point of BC,

so that AD is a median ;

hence the medians of $\triangle$ABC are concurrent and trisect each other. **Q.E.D.**

INDEX

ANSWERS.

NOTE AS TO ANSWERS.

ALL these answers have been calculated (using 4-figure tables and Trigonometry, if necessary), and the results are given to a degree of accuracy at which you might aim. In Geometrical Drawing you will get results varying in reliability according to the intersections involved and to your skill; but too great reliance cannot be put on answers obtained solely by Geometrical Drawing. Frequently the (+) or (−) will give you some clue. [Answer 9·87(+) means more than 9·87, but less than 9·875.] Between ... and ... has been given for results somewhere near midway. In square brackets are given some numbers, which may be useful as checks. In Calculations you should agree exactly or find out where the discrepancy lies.

EXAMPLES 1. (Pages 7 to 13.)

1. 12·7 cm., 2·54 cm. **2.** 3·94″, (i) 0·394″, (ii) 39·4″.
3. AB = 0·98″ = 2·5 cm., PQ = 1·27″ = 3·2 cm.(+), XY = 0·95″ = 2·4 cm.
4. About 0·85 cm.; about 0·75 cm.; former longer.
5. 0·2″(+) = 5½ mm.; always the same.
6. PR = 2·18″, RS = 1·77″, SP = 2·30″. **14.** About 110 yds.
15. Somewhat less than 8·7 cm. **17.** 100 mi.(+).
19. 34 mm. or 34½ mm. **20.** XY is half BC. **28.** About 2·42″.
29. 64°; between 2·74″ and 2·75″. **30.** 15°. **31.** About 2 mi.
32. About 49°, 65½°, 65½°, 114½°, 114½°. **33.** 30°(+).
34. 2·83 mi.(−). **35.** 90°. **37.** 7·1 cm.(−).
38. Equidistant from A, B and C. **40.** 18 ft. 6 in.(−).
41. 153 yds.(−). **42.** About 22 ft. 4 in. **43.** 153 ft.(−) [51 yds.(−)].
44. 10 ft. 9 in. (+). **45.** 54½ ft.(+). **46.** 158 yds.(+).
47. About 12 ft. 10 in. **48.** 0·41 mi.(+). **49.** 880 yds.(+).
50. 95 ft.(+) [32 yds.(−)].

EXAMPLES 2. (Pages 15 to 18.)

1. Between 10·1 cm. and 10·2 cm., 7·6 cm.(+), 5·1 cm.(−).
3. 4·3″(+), 1·9″(+), 3·0″(−). 4. 120 yds.(+). 5. 10·4 cm.(−).
6. 3·4 cm.(+). 7. About 1·91″. 8. 45 mi.(+). 9. 108½ mi.(+).
10. About 30½″. 11. About 224 mi. 12. About 3·85 mi.
13. About 47 mm. 14. 1·44″(−). 15. Just more.
16. 1·6″. 17. 3·9 cm., 3·2 cm., 4·1 cm. 18. 2·64″(+).
21. About 14·1 cm. 22. About 5·07″.

EXAMPLES 3 a. (Pages 25 to 29.)

1. About 47°. 2. About 112½°, 31½°, 36°, 180°. 3. 180°.
4. About 113°+97°+121½°+104°+104½°=540°.
5. About 95°+99°+294°+37°+100°+95°=720°. 6. About 4°.
7. Between 10·58 mi. and 10·59 mi. 8. About 6·35 mi.
9. Between 9·6 mi. and 9·7 mi. 10. About 37°. 11. About 113°.
12. 9 cm., 6 cm. 14. About 61°, etc.
15. Between 6·9 cm. and 7·0 cm., ∠A=about 59½°, ∠B=about 41½°.
16. 0·87″(+), 56°. 17. About 132°, etc. 21. 3·20″(+), 2·83″(−).
22. 10½ ft.(−). 23. About 3·2 cm. or 3·3 cm.
24. About 5·7 cm. or 5·8 cm. 25. About 2·5 cm.
26. Between 4·5 cm. and 4·6 cm., 2·3 cm.(+).
27. About 47 ft., about 32·9 ft., 5·1 ft.(+), 81°(+), 33·3 ft.(−).
28. Bisects it. 29. Rectangle. 31. About 7¾ cm.
32. About 21¼ in. 34. 22½°.
35. Between 2·1 cm. and 2·2 cm., 5·5 cm.(+), 2·3 cm.(+).
36. 7 mi. 37. Between 510 ft. and 520 ft., 350 ft.(−).
38. 19 mm.(+). 39. About 7·2 ft., 680 ft. 40. About 122 yds.
41. About 2″. 42. Returns exactly to his starting point.
43. 368 yds.(−). 44. Between 825 yds. and 830 yds.
45. About 2030 ft. 46. About 215 ft. 47. About 57 ft.
48. About 30¾ ft. 49. About 6¼ in., about 7½°.
50. About 5 ft. 5½ in., about 52°.

EXAMPLES 3 b. (Pages 30 to 32.)

12. 540°. 13. 540°. 14. (1) 9°; (2) 108°; (3) 6480°
15. 180 revs., 1080°. 16. 480°. 17. 7800°
18. 240 revs., 1440°. 19. 528 revs., 108°.
20. $\frac{1056vt}{\pi Td}$ = about 53·8 (taking $\pi = 3\frac{1}{7}$).

EXAMPLES 4 a. (Pages 42 to 48.)

1. 7 mi., just under 9·9 mi. **2.** 3·5 mi., 4·95 mi.(–).
3. About 11·1 mi., 200½°(–).
4. Between 5·1 mi. and 5·2 mi. ; 2·2 mi.(–).
5. 8·2 mi.(–) ; 5·7 mi.(+). **6.** 0·87 mi.(–).
7. 3·5 mi.(+), 20·3 mi.(–). **8.** About 19·6 miles an hour.
9. About 134°, 217½°, 275½°, 350°, 67°.
10. About 151°, 232°, 121°, 262°, 342°, 68°.
11. 1·73 nautical miles(+), 1 nautical mile.
12. Between 3·34 mi. and 3·35 mi., 2·24 mi.(–).
13. (1) 2·39 mi.(–) ; (2) 2·26 mi.(–) ; (3) between 1·95 mi. and 1·96 mi.
14. (1) 10 nautical miles ; (2) 5 nautical miles. **15.** About 53°, 233°.
16. 8·4 cm. or 8·5 cm. **17.** 2·8″ to 2·9″. **18.** 2·4 mi. to 2·5 mi.
19. About 5400 yds. **20.** About 1300 yds. **21.** N. 26° W. (or 334°).
22. 11·3 mi.(+). **23.** 3·89 mi.(–), about 5·07 mi
24. 17·3 mi.(+), 20 mi. **25.** 5 mi. 8°(+).
26. About 82°. **27.** 5·4 mi.(+), about 271°.
28. (i) 7 yds.(–), (ii) about 750 yds., (iii) about ½°.
29. S.E., about 33·3 mi. **30.** S. 43°(+) W. **31.** About 5180 yds
32. 4·0(+) chains N., 8·9(–) chains E.
33. 281 yds.(–) 204°(+). **34.** About 80·4 yds. ; 91°.
35. Between 7·01 mi. and 7·02 mi., 3·47 mi.(–).
36. About 19·1 mi., about 306°. **37.** 0·577 mi.(+), E. 30° S.
38. About 4·75 mi. ; nearer W. than exactly W. by S. [260½°(–)].
39. 111 yds.(–). **40.** (1) 18½°(+) ; (2) 2·6 ft.(–).
41. Between 114 yds. and 115 yds., 41°(+).
42. (1) about 713 yds. ; (2) 62°(–) ; (3) 1½°(–).
43. 3000 yds., 14100 yds.(+). **44.** About 381 ft., 144°(–).

EXAMPLES 4 b. (Pages 48 and 49.)

1. (i) 38° 31′ 22″, 128° 31′ 22″ ; (ii) 77° 25′ 04″, 167° 25′ 04″.
2. 288° 57′ 46″. **3.** About 8·66 mi.
4. N.N.E., E.S.E., W.N.W. **5.** 22½°. **6.** 180°. **7.** 22½°.
9. 30°, 150°, 2100 mi., 10,500 mi. **10.** 47°, 133°. **11.** 2½°, 175 mi.
12. 13° 47′ 39″. **13.** 85° 24′ 41″. **14.** 10 hrs. 04 min. p.m.
15. 50 min. past midnight. **16.** 1 hr. 46 min. 05 sec. p.m.
17. 1 hr. 04 min. 40 sec.
18. 1st August, 4 hr. 30 min. 47 sec. a.m. **19.** 33° 06′ W.

EXAMPLES 5 a. (Pages 60 to 64.)

1. About 0·85″. **2.** About 4 cm. **5.** About 93°, 48½° 93°, 48½°, 38½°.
6. About 2·9 cm. **7.** About 1·45 cm. **8.** About 13½°.

9. About $7\frac{1}{2}°$. **10.** About 68° 83°, 58°, $75\frac{1}{2}°$, $75\frac{1}{2}°$; [correct total 360°]
11. About 113°, 140°, 65°, $107\frac{1}{2}°$, $114\frac{1}{2}°$; [correct total 540°].
12. About $84\frac{1}{2}°$, 81°, −114°, $143\frac{1}{2}°$, $79\frac{1}{2}°$, $85\frac{1}{2}°$; [correct total 360°].
13. About $95\frac{1}{2}° + 99° + 294° + 36\frac{1}{2}° + 100\frac{1}{2}° + 94\frac{1}{2}° = 720°$. **14.** 360°, 900°
15. (i) 0·000,039(+); (ii) $\frac{1}{25344}$. **16.** About 1 : 470,000 [1 : $469,333\frac{1}{3}$].
17. 1 : 50,000. **18.** 1 : 3200. **19.** 5·68″(+), 1 : 1056.
20. About 6·34″ [6·336]. **21.** 4·08″. **22.** 1 : 10,560; 0·34″.
23. 1·59″(+) very nearly; 1 : 31,680; 1 in $18\frac{1}{2}$ about.
24. 3·87″; 3·56″(+) [3·564]; 2·96″(+) [2·961]; 3·96″. **25.** 2·52″.
26. 2·12″(−). **27.** 1·67″(−); 3·33″(+). **28.** About 1·35″.
29. 67°(+). **31.** About 1·75 mi., or about 2·3 mi.

EXAMPLES 5 b. (Pages 64 to 71.)

7. About 4010 mi. **14.** About $40\frac{1}{2}°$, $88\frac{1}{2}°$, 51°; 180°.
15. About $109\frac{1}{2}°$, $63\frac{1}{2}°$, 46°. **16.** 71° 25′. **17.** 2° 41′ 43″.
20. 93°, 56°, 31°. **21.** 54°, 54°, 72°. **22.** 25°, 75°, 100°, 160°.
23. 45°, 75°, 105°, 135°. **25.** 143° each. **26.** $146\frac{1}{4}°$ each.
27. 36. **28.** 110. **29.** 144°, 72°, 144°.
30. 360°. **31.** 540°. **32.** 360°. **33.** 720°.
34. Wrong: 2° too much. **43.** 15 sides. **44.** 180 sides.
45. 18 sides. **46.** 24 sides. **47.** 144°.
48. (i) 156°; (ii) 60. **50.** 3. **54.** 144°; $2\frac{1}{2}$.
56. $2\frac{2}{5}$ sides, *i.e.* 12 sides in 5 circuits. **57.** 60 mi.
58. 132°. **59.** 1080° 00′ 48″, 1080°, −6″, 61° 52′ 53″.

EXAMPLES 5 c. (Pages 72 to 75.)

21. $(180 - \alpha - \beta + \gamma)°$. **24.** 90°. **28.** $\frac{2mn}{m+n}$.

EXAMPLES 6 a. (Pages 81 to 87.)

1. 41°(−). **2.** 5·1 cm.(−) [5·07]. **3.** 2·5″(−).
4. $37\frac{1}{2}°$(−). **5.** About 1·05 ft. **6.** 16 ft. 2 in.(−).
7. About 2·10″, 3·66″, 55°. **8.** About 515 ft. **9.** 3·17 mi.(−).
10. 34°(−). **11.** 28·3 ft.(−). **12.** About 19·6 ft.
13. 2·61″(−), very nearly 1·15″. **14.** 3·65 mi.(−), about 1·08 mi.
16. 112 mi.(−). **17.** 271 yds.(−), 269 yds.(−). **18.** 32 ft.(−).
19. 86 ft.(−). **20.** Between 3·8 cm. and 3·9 cm.
21. $66\frac{1}{2}°$(−), about 27·5 ft., about 37°. **22.** 326 ft.(+).
23. (i) One way; (ii) no ways; (iii) one way; (iv) straight line; (v) shape only; (vi) two ways. **24.** $12\frac{5}{8}$ ft.
25. (1) About 14·9 ft.; (2) about 7 ft. **26.** 44°(−).

27. 8·83 mi.(–). **28.** 31 ft. **29.** About 23·7 ft. ; 35 ft. (very nearly)
30. 11,650 ft.(–) [to nearest 50 ft.]. **31.** 15,600 ft.(+).
32. About 11·3 mi., about 99° or 279°. **33.** About 7·66 mi., 100°.
34. About 290 yds.
35. A little more N. than E. by N. [12°(–)], about E. $13\frac{1}{2}$° S.
36. 18°(+). **37.** About 7·34 ft. **38.** About 2 hrs., 50°(+).

EXAMPLES 7. (Pages 95 to 97.)

4. 2 and central symmetry, 3 and central symmetry.
5. 2 + central symmetry, 1, 6 + central symmetry, central symmetry.
6. 1, 1, 1, 3 and central symmetry.

EXAMPLES 9 a. (Pages 105 to 107.)

8. A = 74°, CA = 4″ ; and (using Prop. 7) AB < 4″. **9.** 60°.
10. 27° each. **11.** 20°, 80°, 80°. **12.** 50°, 65°, 65°.
13. 120°, 30°, 30°. **14.** 152°. **15.** 72°, 72°, 36°.
16. 45°, 45°, 90°. **17.** 108°, 36°, 36°. **18.** 18°, 76°, 86°.
19. (Only one result is given here) $\angle E = 38°$.
20. 6″, 3·6″(+), 4·8″(–). **21.** 80°.
22. (*a*) 72°, each 54°, 108° ; (*b*) 360°, 72°, 108°.
23. 135°. **24.** $64\frac{2}{7}$°. **25.** $73\frac{7}{11}$°.

EXAMPLES 9 b. (Pages 107 to 114.)

12. 60°, 120°. **47.** 15°.

EXAMPLES 10. (Pages 119 to 121.)

31. No. 1 shorter.

EXAMPLES 11. (Pages 127 to 129.)

1. About 7 cm., 11·8 cm. **2.** About 5·5 cm., 12·9 cm.
3. About 106 yds. **4.** 3·16 mi.(+). **5.** About 6·6 in.
6. About 5 ft. 8 in. **14.** About 6·8″. **17.** Between 7·9 cm. and 8·0 cm.

EXAMPLES 12. (Pages 135 to 138.)

1. To $\frac{1}{50}$ of an inch, 2·54″, 1·38″. **2.** $12\frac{5}{8}$ fifths of an inch = $2\frac{21}{40}$ in.
3. 10, 11. **4.** 12·3 mi., 10·8 mi. **5.** $\frac{4}{25}$″, $1\frac{23}{25}$″.
13. Fifths of centimetres. **14.** Eighths of centimetres.
15. 20 mins., 20 secs. **16.** 10 mins., 10 secs.
17. 25 parts, 24 twentieths of an inch.
18. 10 ; 15 [12 + 3] ; 20 [12 + (10 + 1) – 3].
19. One-thousandth of an inch

EXAMPLES 14 a. (Pages 160 to 164.)

1. 12·7 cm.; between 10·1 and 10·2 cm.; about 129 sq. cm.; 6·45 sq. cm.
2. 2·28 in.(–); 10·5 sq. in.
3. 4·60 sq. in.(–).
4. 37° and 143°, about 5·26 sq. in.
5. About 16 sq. cm.
6. About 614 sq. ft.
7. About 5·39 sq. in.
8. 23·0 sq. cm.(–).
9. Between 8·48 and 8·49 sq. in.
10. About 5·42″; 4·37″(–).
11. Very nearly 23·4 sq. cm.
12. Very nearly 44½°.
13. 4·6 cm., about 67°.
14. 35°(–).
15. 51½°(–).
16. Between 3·15 and 3·16 sq. in.
17. 7·5 or 7·6 sq. cm.
18. Between 7·60 and 7·61 chains, very nearly 2·1 acres.
19. 3·2″.
20. 76°(+).
21. 14·1 sq. cm.(–).
22. A = 65°(+), C = 48°(–), b = 8·1 cm.(+), c = 6·5 cm.(+).
23. 71°(–).
24. 42°(–), 138°(+), 90°.
25. 60 sq. cm., 180 sq. cm.
26. About 24½ sq. cm.
27. 2·5″, 115°(+).
28. About 2[illegible]½ sq. cm.
29. 2·36 sq. in.(–).
30. 4 sq. in.(–).
31. Between 13,000 and 13,100 sq. yds.
32. 7·7 sq. cm.(+).
33. Between 407 yds. and 408 yds.; 12·9 acres(–).
34. 36,160 sq. yds.(–) or about 7·47 acres.
35. 330 yds.(–); about 58,450 sq. yds. = about 12 acres.
36. About 4·87 in.
37. Between 4·1 cm. and 4·2 cm.
38. About 25 sq. cm.
39. About 29·5 sq. cm.
41. About 36·7 sq. cm.
42. About 9·24 sq. in.
43. 31·4 yds.(–).

EXAMPLES 14 b. (Pages 165 to 173.)

1. 11 sq. ft. 36 sq. in.
2. 225 sq. in.
3. 441 sq. in.
4. 0·3 sq. in.
5. 275 sq. yds., 90 yds., 80 yds.
6. 74 yds., 76 yds., 122 sq. yds.
7. 240 sq. in.
8. 21 sq. in.
9. 11 ft. 3 in.
10. 3·5″(–) $\left[\frac{10\cdot4}{3}\right]$.
11. 13 sq. in.
12. 25 sq. in.
13. 12 sq. in.
14. 15 sq. ft.
15. 1·04 sq. cm.
16. 47·6 sq. cm.
17. 6 sq. in.
18. 1·32 sq. cm.
19. 42 sq. in.
20. Between 3·3 cm. and 3·4 cm., over 12 sq. cm.
21. 43 quarter-inch squares = $2\frac{11}{16}$ sq. in.
22. 13, 8, 4; 75 sq. cm.
23. 3″.
24. 4 sq. ft. 13 sq. in.
25. 6·52 sq. in.
26. 260 sq. mm.
27. 405 sq. in.
28. About 19·19 acres, 9·29 sq. in.(–).
29. 15·96 sq. cm.
30. 7 sq. in.
31. 1·17 m.(–).
32. 53.
33. 10 cm.
34. 160 sq. ft.; about 57·7 cu. ft.
35. ..., ..., $y = \frac{3}{2}x(8 - x)$, maximum when $x = 4$.
36. About 22½ sq. ft., 27¼ sq. ft., 8 sq. ft., 19⅓ sq. ft. [$22\frac{1}{16}$, $27\frac{9}{10}$, $7\frac{7}{8}$, $19\frac{7}{16}$].
37. (b) $3600x + 75x^2$ cu. ft.; (d) 16·8 ft.
38. 2·459 acres(–).
39. 0·6 sq. in.
40. 3′ 9″.
41. 256½ sq. ft., 4120 tons.

42. 6·79 cm^2. 43. $766\frac{1}{2}$ sq. ft.(–). 44. 1407 sq. ft.(–).
46. 4·064 acres. 47. 7·276 acres. 48. 8·772 Ha.
49. $1\frac{1}{70}$ sq. in. 50. 308 sq. units. 51. $51\frac{1}{2}$ sq. mi.
52. 7·24″(–) and 2·76″(+) [$5 \pm \sqrt{5}$].

EXAMPLES 15 a. (Pages 188 to 200.)

1. 17 ft. 2. 144 yds.(+). 3. 25 ft.
4. 25 ft. 5. 13 ft. 6. $\sqrt{d^2+(y-x)^2}$ ft.
7. 9·90 mi. (very nearly), 7 mi. (exactly). 8. (i) 1·7″, (ii) 2·4 sq. in.
9. 330 sq. in. 10. 2·5 cm., 7·5 sq. cm., 2·31 cm.(–) [$2\frac{4}{13}$].
11. 40 ft. 12. 2·4″. 13. 450 yds., 126,000 sq. yds.
14. $114\frac{3}{4}$ yds.(+), about $2008\frac{2}{3}$ sq. yds. 15. 210 sq. in. each.
16. 6 yds., about 5 yds. $10\frac{1}{2}$ in. 17. Yes [shortest diagonal of brick is 5·5″].
18. About 26 yds. 2 ft. 19. 2 hrs. 16 min.
20. A $4\frac{1}{2}$ mi., B 3 mi., or A $13\frac{1}{2}$ mi., B 9 mi.
21. ... , middle points of sides, 0·79″(–) or 6·21″(+).
22. A little over 76 ft. [26 + 15 + 35·11...]. 23. 9 ft., 17 ft. 8 in.(–).
24. 20 ft. 5 in.(–). 25. 6 or 15. 26. 23 ft.
27. 32 ft. 6 in.(–). 28. 15 ft. 7 in.(+).
29. 14 ft., very nearly 47 ft. $4\frac{1}{2}$ in.(–). 30. 17·64 Ha.
31. Nearly 9·8 cm.
32. Parallelogram, rectangle, kite, isosceles triangle.
33. 45°, 37°(–), 98°(+). 34. 77°(–), 13°(+), 17·46″(+), 5″.
35. 39″. 36. 9″, 82·24 sq. in. 37. 65″.
38. AB = BC = 1·84″(–) [$1·3 \times \sqrt{2}$]; CD = 2·4″, BM = 1·3″, DN = $1\frac{2}{3}$″; 50 : 119 : 169.
39. 10 mi., 10 mi., 14·1 mi.(+) [$10\sqrt{2}$]. 40. 29 mi.
41. 1·62 mi.(–) [$\sqrt{2·61}$]. 42. 2·8 mi. 43. $\sqrt{(a-c)^2+(b-d)^2}$ mi.
44. 5·66 cm.(–) [$4\sqrt{2}$]; 56·6 sq. cm.(–) [$40\sqrt{2}$]. 45. 30 ft., 816 sq. ft.
46. 6·7 acres(–). 47. 8 cm., 11·2 cm.(–). 48. 124 cm., 710 sq. cm.
49. 8·54 sq. in. 50. 15 sq. in. 51. 19·0 sq. in.(–) [$6\sqrt{10}$].
52. 1·6 in., 4·8 sq. in. 53. 7·4 cm., about 22·2 sq. cm.
54. $p^2(3-2\sqrt{2})$. 56. Greatest area = 25 sq. ft.
57. 6 cm., 10·39 cm.(+); 31·2 sq. cm.(–). 58. 1·73″(+) [$\sqrt{3}$].
61. About 60·6 sq. cm. 62. Between 21·2 and 21·3 cm.
63. Between 8·48 sq. in. and 8·49 sq. in. 65. $5\frac{3}{4}$ cm.
66. 29·7 in.(–) [$21\sqrt{2}$); 9·9 in.(–) [$7\sqrt{2}$].
67. 2·6 sq. cm.(–) [$\frac{3}{2}\sqrt{3}$]; over 33 thousand.
68. About 21·65 sq. yds. [$\frac{25}{2}\sqrt{3}$]; 223 or 224.
69. 26, 76, 51 hundreds of each kind. 70. Area = 126 sq. in.
71. 10·8 sq. cm.(+); 31·2 sq. cm.(–). 72. 1·5 in., 4·2 sq. in.
73. 22 ft. 4 in.(+); about 23 ft. 9 in. 74. 25 ft., 27 ft. 4 in.(–).

75. 173 cm **76.** 13 ft., 15 ft., about 10·3 ft., $42\frac{1}{2}°(-)$.
77. $\sqrt{a^2+b^2}$ ft., $\sqrt{a^2+b^2+c^2}$ ft. **78.** 200 cu. cm.; 332·5 sq. cm.
79. 26 ft., 11 ft. 7 in.$(-)$ [$\sqrt{134}$]. **80.** 1·73″$(-)$, 1·95″$(+)$, 1·30″$(+)$.
81. Over 55 ft. 3 in. **82.** YZ=2·14″$(-)$; ZX=1·61″$(+)$; XY=2·42″$(-)$.
83. About 11·36″. **84.** About 14·72″. **85.** About 12′ 8″.
86. $\dfrac{k}{3\sqrt{a^2+b^2+c^2}}\{\sqrt{a^2+(b+c)^2}+\sqrt{b^2+(c+a)^2}+\sqrt{c^2+(a+b)^2}\}$ ft. per sec.

EXAMPLES 16 a. (Pages 212 to 216.)

1. Between 1·17″ and 1·18″. **2.** About 3·60″.
3. Between 1·40″ and 1·41″.
4. Between 4·7 cm. and 4·8 cm. twice or 1·6 cm.$(-)$ twice.
5. 0·40″$(-)$, or between 1·09″ and 1·10″.
6. Between 1·5 mi. and 1·6 mi. or between 2·2 mi. and 2·3 mi.
7. 5 solutions. **8.** Between 9·5 cm. and 9·6 cm. or 16·8 cm.$(-)$.
14. About 20 cm. **16.** 3·1 cm.$(+)$. **17.** 8 cm.
18. 39°$(-)$, 16″, between 13·6″ and 13·7″.
25. (i) 3·97″$(-)$; (ii) $3\frac{1}{2}$″ exactly.

EXAMPLES 17. (Pages 221 to 223.)

1. 4. **2.** 3, 2. **3.** All the same. **5.** 0.
6. 22. **7.** An infinite number. **8.** 0.
9. 4. **10.** 5, 3. **11.** 4 [fix opposite, not adjacent, corners].
13. 2, 3, 6. **14.** 5, 1, 17·64 sq. cm. (exact). **15.** 7.

Pricking off.

Constant pricking off wears out the exact points. It is not a bad plan to prick off the points very occasionally, on to a loose piece of paper, to keep that piece of paper in the book, and to use that piece of paper. It has the additional advantage of allowing you to prick off your figure in the position on your paper most convenient to you.

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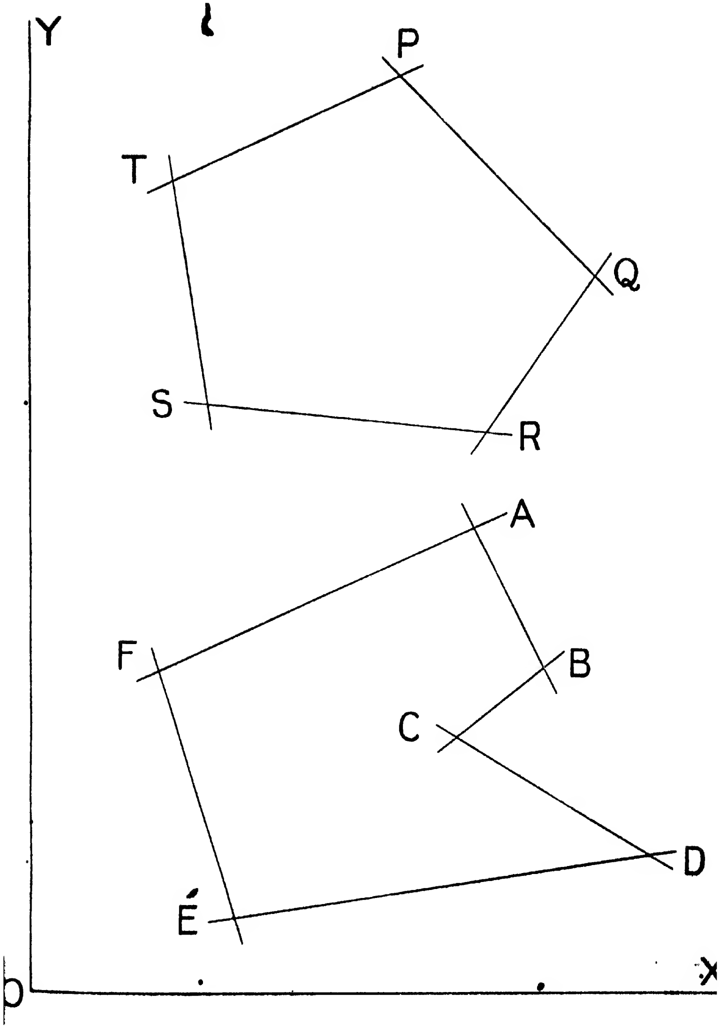

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